Theoretical Physics V

SS 2014 **16.4.2014**
Assignment I **Due date 23.4.2014**

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Problem 1 *Stationary state perturbation theory*

a) Consider a hydrogen-like atom where an electron has the potential energy

$$V(r) = \begin{cases} -\frac{Ze^2}{R}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R^2}\right), & 0 \le r \le R \\ -\frac{Ze^2}{r}, & r \ge R, \end{cases} \tag{1}$$

with R the radius of the nucleus and Ze the nuclear charge. When the nucleus is assumed to be point-like, the electron ground state wave function is

$$\psi_0(x) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \quad a = \frac{\hbar^2}{Zme^2} \tag{2}$$

and the ground state energy is $E_0^{(0)} = -\frac{Ze^2}{2a}$. Within the first order stationary state perturbation theory calculate the ground state energy E_0 of the atom with a finite radius. How does the first order correction in the ground state energy scale with Z? *(1.5 points)*
Hint: The radius of the nucleus is much smaller than the Bohr radius a.

b) Consider an oscillator with the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2 x^2}{2} + c_1 x^3 + c_2 x^4. \tag{3}$$

With $c_1 = c_2 = 0$ this Hamiltonian describes a harmonic oscillator with the ground state energy $E_0^{(0)} = \frac{\hbar\omega}{2}$ and the ground state wave function

$$\psi_0(x) = (x_0\sqrt{\pi})^{-1/2} e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}, \quad x_0 = \sqrt{\frac{\hbar}{m\omega}}. \tag{4}$$

Find the ground state energy for an anharmonic oscillator where c_1 and c_2 are small but finite using the first order stationary state perturbation theory. *(1 point)*

Problem 2 *Time-independent degenerate perturbation theory*

The Hamiltonian describing an isotropic harmonic oscillator in two dimensions is given by

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2\right) \tag{5}$$

a) What are the eigenenergies of the three lowest-lying eigenstates? Is there any degeneracy? *(1 point)*

b) Consider a interaction Hamiltonian $V = \delta m\omega^2 xy$ where δ is a dimensionless number and $\delta \ll 1$. Find the zeroth-order correction to the eigenstates and the corresponding energy shift to first order for the states above. *(1 point)*

c) Now find the eigenenergies exactly and compare with the results from perturbation theory. *(1 point)*

Problem 3 *Quasi-degenerate perturbation theory*

Consider a 3-level atom in a Λ configuration, driven by two off-resonance laser fields (see Fig.1). The Hamiltonian for this system, in the rotating wave approximation is given by $H = H_0 + V$, where

$$H_0 = \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \sum_{i=1}^{3} e_i \left|i\right\rangle \left\langle i\right| , \tag{6}$$

$$V = \hbar g_a \left(\left|3\right\rangle \left\langle 1\right| a + \left|1\right\rangle \left\langle 3\right| a^\dagger\right) + \hbar g_b \left(\left|3\right\rangle \left\langle 2\right| b + \left|2\right\rangle \left\langle 3\right| b^\dagger\right) . \tag{7}$$

Here ω_i is the frequency of laser $i = a, b$ with creation and annihilation operators $a^\dagger(b^\dagger)$ and $a(b)$, e_j are the energies of the atomic levels $j = 1, 2, 3$ and g_a, g_b the corresponding coupling constants.

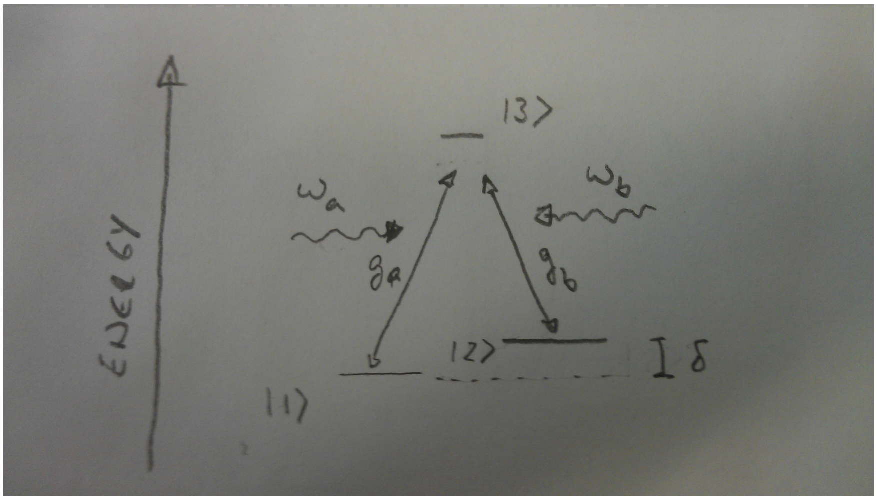

Abbildung 1: Energy scheme for two lasers driving an atom in a Λ configuration.

a) What are the eigenstates and eigenenergies of Hamiltonian H_0? What are the energy differences between the eigenstates of H_0 connected by V? *(1 point)*

b) If g_a and g_b are much smaller than the detuning of the drive lasers to the atomic transitions, i.e. $(e_3 - e_1)/\hbar - \omega_a$ and $(e_3 - e_2)/\hbar - \omega_b$, we can threat V as a perturbation. In this regime use the quasi-degenerate perturbation and find the unitary matrix that decouples the low- and high-energy subspaces of H_0. *(2 point)*

c) Show that the resulting effective Hamiltonian has shifted laser frequencies (known as AC-Stark shift) and shifted atomic level frequencies (Lamb shift). To first approximation transitions to the excited atomic level are no longer driven. Interpret the term coupling the two low-energy atomic levels. *(2 point)*

Problem 4 *Time-dependent perturbation theory*

a) Show that if the observables $\hat{A}$, $\hat{B}$, and $\hat{C}$ satisfy the commutation relation $[\hat{A}_S, \hat{B}_S] = i\hat{C}_S$ in the Schrödinger picture, they satisfy the same relation also in the Heisenberg and interaction pictures. *(0.5 points)*

b) Show that for the harmonic oscillator with the Hamiltonian operator

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2 x^2}{2} \tag{8}$$

the position operator in the Heisenberg picture is

$$\hat{x}_H(t) = \hat{x}_S \cos(\omega t) + \frac{\hat{p}_S}{m\omega}\sin(\omega t), \tag{9}$$

with $\hat{x}_S$ and $\hat{p}_S = -i\hbar\frac{\partial}{\partial x}$ the position and momentum operators in the Schrödinger picture, respectively. *(2 points)*
Hint: Expand $\hat{x}_H(t)$ in the Taylor seriers $\hat{x}_H(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\left[\left(\frac{d}{dt}\right)^n \hat{x}_{H|t=0}\right]$ and use Heisenberg equation of motion to calculate the time derivatives. You need to evaluate the commutators $[\hat{H}, \hat{x}_S]$, $[\hat{H}, [\hat{H}, \hat{x}_S]]$, $[\hat{H}, [\hat{H}, [\hat{H}, \hat{x}_S]]]$ etc.

c) A charged particle (charge q) is in harmonic potential where time-dependent electric field is applied

$$\hat{H} = \hat{H}_0 + V(t),$$
$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2 x^2}{2}, \quad V(t) = -xqE(t), \ E(t) = \frac{C}{\sqrt{\pi}\tau}e^{-(t/\tau)^2}. \tag{10}$$

Here C and τ are constants. At time $t = -\infty$ the oscillator is in its ground state. Within the first order time-dependent perturbation theory, find the probability that at time $t = +\infty$ the oscillator will be in its first excited state. *(3 points)*
Hint: In the absence of the electric field the wave functions of the ground state and the first excited state (energy $E_1 = \frac{3\hbar\omega}{2}$) are

$$\psi_0(x) = (x_0\sqrt{\pi})^{-1/2} e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}, \quad \psi_1(x) = \frac{\sqrt{2}}{\pi^{1/4} x_0^{3/2}} x e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}, \quad x_0 = \sqrt{\frac{\hbar}{m\omega}}. \tag{11}$$