Theoretical Physics V

SS 2014 **18.6.2014**
Assignment X **Due date 25.6.2014**

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Problem 1 *Josephson current*

Brian Josephson received the Nobel Prize in physics in 1973 for his theoretical predictions of the properties of a supercurrent through a tunnel barrier, in particular those phenomena generally known as the Josephson effects. Let us study the microscopic mechanism of the DC Josephson effect, a direct current crossing a tunneling barrier without any external electric field. A Josephson junction consists of two superconductors, separated by an insulating layer through which the electrons can tunnel. Its Hamiltonian $\hat{H}$ can be expressed as a sum

$$\hat{H} = \hat{H}_L + \hat{H}_R + \hat{H}_T. \tag{1}$$

Here $\hat{H}_L$ ($\hat{H}_R$) is the reduced Hamiltonian (cf. Exercise VIII/1) describing the superconductor on the left (right) hand side of the insulating barrier in the absence of coupling between the two superconductors. The electron tunneling through the barrier can be treated as a perturbation

$$\hat{H}_T = \sum_{\mathbf{r},\mathbf{l},\sigma} \left(T_{\mathbf{r},\mathbf{l}} a^\dagger_{\mathbf{r}\sigma} a_{\mathbf{l}\sigma} + T^*_{\mathbf{r},\mathbf{l}} a^\dagger_{\mathbf{l}\sigma} a_{\mathbf{r}\sigma} \right). \tag{2}$$

Here the states $\mathbf{r}$ and $\mathbf{l}$ belong to the superconductors on the right and left sides, respectively. The matrix element $T_{\mathbf{r},\mathbf{l}}$ can be calculated from Schrödinger equation but is here assumed to be given.
Let us expand the total wave function with N electrons ($N/2$ pairs) in the form

$$|\Psi_N\rangle = \sum_n b_n |\Psi_{N-n}\rangle_L |\Psi_n\rangle_R, \tag{3}$$

where $|\Psi_n\rangle_R$ describes n electrons on the right side.
By second order perturbation theory it can be shown for the eigenenergy E of $\hat{H}$ that

$$E b_{2n} = 2E_F b_{2n} - \frac{E_J}{2}[b_{2(n-1)} + b_{2(n+1)}], \tag{4}$$

where E_J is called Josephson energy. The equation (4) is formally equivalent to describing the motion of a particle in a chain of atoms. The analogue of the coefficient b_{2n} is an atomic orbital localized around the nth atom. In the tight binding approximation one tries to find the wave function as a linear combination of localized orbitals.

a) Show that $b^{(\varphi)}_{2n} \sim e^{i\varphi n}$ is the solution of (4) and calculate the corresponding eigenenergy $E = E(\varphi)$. The variable φ is referred as the phase difference across the junction. In the analogue of the atomic chain, what does this state describe?

(0.5 points)

b) Suppose we construct a wave packet of the $b_{2n}^{(\varphi)}$s corresponding to phases between φ and $\varphi + \Delta\varphi$. Then by the analogy with the linear atomic chain, the spatial extent of the wave packet is $\Delta n \sim \frac{1}{\Delta\varphi}$. Since $n \sim 10^{22}$ and $\Delta n \sim \sqrt{n} \sim 10^{11}$ are very large it is possible to have both $\Delta\varphi$ and $\frac{\Delta n}{n}$ very small and specify both n and φ simultaneously very precisely. The rate of change of the number of electron pairs in the right superconductor can be obtained through the group velocity of the wave packet

$$\frac{d\langle n\rangle}{dt} = \frac{1}{\hbar}\frac{\partial E(\varphi)}{\partial\varphi}. \tag{5}$$

The supercurrent through the junction

$$I(\varphi) = 2e\frac{d\langle n\rangle}{dt} \tag{6}$$

is called the *Josephson current*. Calculate $I(\varphi)$. You can assume E_J is given. *(0.5 points)*

c) Let us now denote the eigenstate of the bare reduced Hamiltonian by $|\Psi^{(0)}\rangle$, the superscript indicating the state is not affected by perturbations such as $\hat{H}_T$. Let us denote the states where in $|\Psi^{(0)}\rangle$ one pair of electrons has been replaced by an unpaired electron by

$$|\Psi^{(0)}_{\tilde{\mathbf{k}},\alpha}\rangle \equiv \prod_{\mathbf{k}\neq\tilde{\mathbf{k}}}(u_{\mathbf{k}} + v_{\mathbf{k}}a^\dagger_{\mathbf{k},\uparrow}a^\dagger_{-\mathbf{k},\downarrow})a^\dagger_{\tilde{\mathbf{k}},\alpha}|0\rangle. \tag{7}$$

The *quasiparticle* operators can be defined through *canonical transformation*

$$\gamma^\dagger_{\mathbf{k}\uparrow} \equiv u_{\mathbf{k}}a^\dagger_{\mathbf{k}\uparrow} - v_{\mathbf{k}}a_{-\mathbf{k}\downarrow}, \quad \gamma^\dagger_{\mathbf{k}\downarrow} \equiv u_{\mathbf{k}}a^\dagger_{\mathbf{k}\downarrow} + v_{\mathbf{k}}a_{-\mathbf{k}\uparrow} \tag{8}$$

that leaves the canonical commutation relations invariant. Show that

$$|\Psi^{(0)}_{\mathbf{k},\alpha}\rangle = \gamma^\dagger_{\mathbf{k}\alpha}|\Psi^{(0)}\rangle, \quad \gamma_{\mathbf{k}\alpha}|\Psi^{(0)}\rangle = 0. \tag{9}$$

(1 point)

d) The Josephson energy

$$E_J = -4\sum_{\mathbf{r},\mathbf{l},\alpha,\beta}\frac{\langle 2(n+1)^{(R)}|\hat{H}_T|V_{\mathbf{r}\alpha,\mathbf{l}\beta}\rangle\langle V_{\mathbf{r}\alpha,\mathbf{l}\beta}|\hat{H}_T|2n^{(R)}\rangle}{E_{\mathbf{r}} + E_{\mathbf{l}}} \tag{10}$$

is due to processes in which two electrons are transferred across the junction via a virtual state $|V_{\mathbf{r}\alpha,\mathbf{l}\beta}\rangle$ that has a single electron transferred through the junction. In the virtual state the corresponding unpaired electrons on the right and left have quasiparticle energies $E_{\mathbf{r}}$ and $E_{\mathbf{l}}$. The initial and final states in the process are $|2n^{(R)}\rangle \equiv |\Psi^{(0)}_{2n}\rangle_R|\Psi^{(0)}_{2N-2n}\rangle_L$ and $|2(n+1)^{(R)}\rangle \equiv |\Psi^{(0)}_{2n+2}\rangle_R|\Psi^{(0)}_{2N-2n-2}\rangle_L$, respectively. Let $|\Psi^{(0)}_{2n,\mathbf{r}\alpha}\rangle_R$ be the normalized component of $|\Psi^{(0)}_{\mathbf{r},\alpha}\rangle_R$ with $2n+1$ electrons on the right. Then the virtual state is $|V_{\mathbf{r}\alpha,\mathbf{l}\beta}\rangle \equiv |\Psi^{(0)}_{2n,\mathbf{r}\alpha}\rangle_R|\Psi^{(0)}_{2N-2n-2,\mathbf{l}\beta}\rangle_L$. Show that

$$E_J = 8\sum_{\mathbf{r},\mathbf{l}}|T_{\mathbf{r},\mathbf{l}}|^2\frac{u_{\mathbf{r}}v_{\mathbf{r}}u_{\mathbf{l}}v_{\mathbf{l}}}{E_{\mathbf{r}} + E_{\mathbf{l}}}. \tag{11}$$

(4 points)

Hint: Use the results of Exercise IX/1.

Problem 2 *4-notation and Lorentz transformations*

a) In Classical mechanics, Newtons 2nd law reads $\vec{F} = m\vec{a}$. Show that the relativistic equation $\vec{F} = d\vec{p}/dt$ gives

$$\vec{F} = \frac{m}{\sqrt{1 - u^2/c^2}} \left(\vec{a} + \frac{\vec{u}(\vec{u} \cdot \vec{a})}{c^2 - u^2} \right) \tag{12}$$

where $\vec{a} = d\vec{u}/dt$ is the ordinary acceleration and $\vec{u}$ the velocity. *(1 points)*

b) Define the proper acceleration as

$$\alpha^\mu \equiv \frac{d\eta^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2}\,, \tag{13}$$

with η^μ the 4-velocity and τ the proper time. Find α^0 and $\vec{\alpha}$ in terms of $\vec{a}$ and $\vec{u}$. *(1 points)*

c) Express $\alpha_\mu \alpha^\mu$ in terms of $\vec{a}$ and $\vec{u}$ and show that $\eta^\mu \alpha_\mu = 0$. *(1 points)*

d) Write the Minkowski version of Newton's second law in terms of α^μ and evaluate the inner product $K^\mu \eta_\mu$, where $K^\mu \equiv dp^\mu/d\tau$ is the Minkowski 4-force. *(1 points)*

e) Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

$$v = \frac{v_1 + v_2}{1 + (v_1 v_2/c^2)}. \tag{14}$$

This is an alternative way to derive the parallel-velocity addition law. *(2 point)*