Theoretical Physics V

SS 2014 **25.6.2014**
Assignment XI **Due date 2.7.2014**

Prof. Dr. Wilhelm-Mauch http://qsolid.uni-saarland.de/?Lehre:TP_V

Problem 1 *Dirac gamma matrices*

Let us choose the following representation for the Dirac gamma matrices

$$\gamma^0 \equiv \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \; \gamma^i \equiv \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{1}$$

with $\sigma_{1,2,3}$ the Pauli matrices and σ_0 a 2×2 unit matrix.

a) Show that the representation above satisfies the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2}$$

for the γ matrices. Here $g^{\mu\nu}$ is the metric tensor in flat Minkowski space. *(0.5 points)*

b) Calculate the anticommutator $\{\gamma^\mu, \gamma^5\}$ where

$$\gamma^5 \equiv \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \; \mu \in \{0, 1, 2, 3\}. \tag{3}$$

(0.25 points)

c) Calculate $\gamma^\mu \gamma_\mu$. We use Einstein's summation convention, *i.e.*, assume a sum over $\mu \in \{0, 1, 2, 3\}$. *(0.5 points)*

d) Calculate $\gamma^\mu \not{a} \gamma_\mu$. We use Feynman slash notation, *i.e.*, $\not{a} \equiv a^\mu \gamma_\mu$. *(0.75 points)*

e) Calculate $\gamma^\mu \not{a} \not{b} \gamma_\mu$. *(1.25 points)*

f) Calculate $\mathrm{Tr}[\not{a} \not{b}]$. *(1 point)*

g) Show that $\mathrm{Tr}[\not{a} \not{b} \not{c} \not{d}] = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$. Here $a \cdot b \equiv g_{\mu\nu} a^\mu b^\nu$. *(1.75 points)*

h) Calculate $\mathrm{Tr}[\gamma_2 \gamma_1 \gamma_3 \gamma_1 \gamma_2 \gamma_3 \gamma_1 \gamma_2 \gamma_1 \gamma_2 \gamma_3]$. *(1 point)*

Problem 2 *Dirac equation and wave-packets*

Since the Dirac equation is linear, the plane-wave solutions found in class can be superposed to form localized wave-packets. We shall study them in this exercise.

a) First, show that, for any two solutions $\psi_1(x)$ and $\psi_2(x)$ of the Dirac equation $(p - mc)\psi(x) = 0$,

$$c\bar{\psi}_2\gamma^\mu\psi_1(x) = \frac{1}{2m}\left[\bar{\psi}_2 p^\mu \psi_1 - (p^\mu\bar{\psi}_2)\psi_1\right] - \frac{i}{2m}p_\nu(\bar{\psi}_2\sigma^{\mu\nu}\psi_1) \quad (4)$$

(1 point)

b) Construct a (normalized) wave-packet $\psi^{(+)}(x,t)$ by superposing positive-energy solutions ONLY and use the above equation to show that the average current $J^{(+)} = \int d^3x \psi^{(+)\dagger} c\alpha\psi^{(+)}$ for this positive-energy wave-packet is just the classical group velocity $\langle c^2 p/E\rangle_+$. Here $<>_+$ denotes the expectation value with respect to a positive-energy wave-packet.
(1 point)

c) We can now realize an important difference in the relativistic theory. In the Schrdinger theory, the velocity appearing in the current is just p/m and is a constant of motion for free particles. The Dirac current is however not proportional to the momentum and the velocity operator $c\alpha$ is not a constant of motion, since $[\alpha, H] \neq 0$. Let us now include both positive and negative energy solutions in the wave-packet. Find the conditions for the wave-packet to be normalize to unit and show that, in addition to the time-independent group velocity, the current now also contain cross terms between the positive- and negative-energy solutions, oscillating rapidly in time.
(2 point)

d) The general form of the free-particle solution obtained above shows that a wave-packet formed initially of positive-energy solutions only, will NOT develop negative-energy solutions during the dynamics. However, a wave-packet representing an electron initially localized in a finite region in space, usually include both positive- and negative-energy solutions. Consider a packet which at $t = 0$ has a Gaussian distribution with half-width d. Match the solution found above to this wave-packet and determine the expansion coefficients for the different energy solutions. Show that both positive- and negative-energy coefficients are non-zero. *Hint:* Use the free-particle orthogonality relations.
(2 point)