# Theoretical Physics V

**SS 2014** **2.7.2014**
**Assignment XII** **Due date 9.7.2014**

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## Problem 1 *Charge conjugation*

Let $\psi$ be a solution of the Dirac equation

$$i\frac{\partial\psi}{\partial t} = \left[\alpha_i\left(-i\frac{\partial}{\partial x_i} - eA^i\right) + eA^0 + \beta m\right]\psi. \tag{1}$$

Here

$$\beta \equiv \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix},\ \alpha_i \equiv \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \tag{2}$$

with $\sigma_{1,2,3}$ the Pauli matrices and $\sigma_0$ a $2\times 2$ unit matrix. Furthermore, $A^0 = \phi$ is the scalar potential and $A^i$ are the components of the vector potential.

a) Show that the *charge conjugated* state $\psi^C \equiv C\beta\psi^*$ with

$$C = \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \tag{3}$$

satisfies the Dirac equation with the opposite charge

$$i\frac{\partial\psi^C}{\partial t} = \left[\alpha_i\left(-i\frac{\partial}{\partial x_i} + eA^i\right) - eA^0 + \beta m\right]\psi^C. \tag{4}$$

Above the asterisk denotes complex conjugation. *(1.5 points)*

b) Show that the free particle Dirac equation is satisfied by the normalized positive and negative energy solutions

$$\psi^{(+)}_{\mathbf{p},s}(x) = \frac{1}{\sqrt{2E_pL^3}}u(\mathbf{p},s)e^{-i\mathbf{p}\cdot\mathbf{x}}, \quad \psi^{(-)}_{-\mathbf{p},-s}(x) = \frac{1}{\sqrt{2E_pL^3}}v(\mathbf{p},s)e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{5}$$

where positive and negative energy Dirac spinors are

$$u(\mathbf{p},s) \equiv \sqrt{E_p+m}\begin{pmatrix} 1 \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p+m} \end{pmatrix}\chi^{(s)}, \quad v(\mathbf{p},s) \equiv \sqrt{E_p+m}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p+m} \\ 1 \end{pmatrix}[-i\sigma_2\chi^{(s)}]. \tag{6}$$

Here the spinors $\chi^{(s)}$ are eigenvectors of $\sigma_3$

$$\chi^{(\frac{1}{2})} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi^{(-\frac{1}{2})} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7}$$

*(2 points)*

c) Above $\psi^{(+)}_{\mathbf{p},s}(x)$ is the wave function of a particle with positive energy, momentum $\mathbf{p}$, and spin $s$, whereas $\psi^{(-)}_{-\mathbf{p},-s}(x)$ is the wave function of a particle with negative energy, momentum -$\mathbf{p}$, and spin $-s$. Charge conjugation provides a relation between negative energy and antiparticle states. The state $\psi^{(-)C}$ can be identified with the physical positive energy state corresponding to $\psi^{(-)}$. Show that $[\psi^{(-)}_{-\mathbf{p},-s}(x)]^C = \psi^{(+)}_{\mathbf{p},s}(x)$. *(1.5 points)*

## Problem 2 *Useful properties of the solutions of the Dirac equation*

We already know what are the solutions of Dirac's equation for a free particle at rest.

$$\psi^r(x) = w^r(0)e^{-(i\epsilon_r mc^2/\hbar)t},$$

with $r = 1, 2, 3, 4$ and $\epsilon_1 = \epsilon_2 = +1$ and $\epsilon_3 = \epsilon_4 = -1$. To find the solutions for a free particle with an arbitrary velocity, we can make use of the Lorentz transformations. The basic idea is to use this transformation to go to a coordinate system moving with a velocity $-\mathbf{v}$ relative to the solutions at rest, which correspond to an observed free-particle with velocity $+\mathbf{v}$.

a) How does the exponential function in $\psi^r$ transform? *(1 point)*

b) Take the simple case where $\mathbf{v}$ is in the $x$-direction. Write down in matrix form the transformation matrix for the spinors $w^r$. Express your final result in terms of the energy and momentum of the particle. *(1 point)*

c) Generalize the previous result for a velocity in an arbitrary direction. *(1 point)*

d) Prove the following useful identities:

$$(\not{p} - \epsilon_r mc)w^r(\mathbf{p}) = 0$$
$$\bar{w}_r(\mathbf{p})(\not{p} - \epsilon_r mc) = 0$$
$$\bar{w}^r(\mathbf{p})w^{r'}(\mathbf{p}) = \delta_{rr'}\epsilon_r$$
$$\sum_{r=1}^{4} \epsilon_r w^r_\alpha(\mathbf{p})\bar{w}^r_\beta(\mathbf{p}) = \delta_{\alpha\beta}$$

*(2 point)*

e) It is often convenient to have operators that project out spinors of a given sign in energy. This operator should be given in a covariant form and satisfy the usual properties for projectors, i.e.,

$$P_r(\mathbf{p})w^{r'}(\mathbf{p}) = \delta_{rr'}w^{r'}(\mathbf{p})$$
$$P_r(\mathbf{p})P_{r'}(\mathbf{p}) = \delta_{rr'}P_r(\mathbf{p}).$$

Verify that the operator

$$\Lambda_r(p) = \frac{\epsilon_r \not{p} + mc}{2mc}$$

satisfy this conditions. *(1 point)*