Theoretical Physics V

SS 2014 **30.4.2014**
Assignment III **Due date 7.5.2014**

Prof. Dr. Wilhelm-Mauch http://qsolid.uni-saarland.de/?Lehre:TP_V

Problem 1 *Time-dependent perturbation theory: Energy scales*

An electron (charge $-e$) is at time $t = -\infty$ in the stationary state $\langle \mathbf{r}|n\rangle = \psi_n(x, y, z)$ bound to an atom. Let the atom be centered at $\mathbf{R}_a = (0, 0, 0)$. A heavy charged particle (charge Ze) with velocity $v\mathbf{e}_x$ passes near the atom so that its trajectory can be described by $\mathbf{R}_p(t) = (vt, d, 0)$. Here the distance of nearest approach to the atom d is much larger than the size of the atom.

a) Find the probability that the electron will make a transition from the state $|n\rangle$ to the unoccupied bound state $|m\rangle$ due the influence of the heavy particle. You can assume that the energies E_n and E_m of the states $|n\rangle$ and $|m\rangle$ as well as the matrix elements $\langle n|x|m\rangle$, $\langle n|y|m\rangle$, and $\langle n|z|m\rangle$ are given. You do not need to evaluate the time integral arising from perturbation theory in this part. *(1.5 points)*
Hint: Use first order time-dependent perturbation theory to calculate the probability that at time $t = \infty$ the electron will be in the state $|m\rangle$. Expand $\frac{1}{|\mathbf{R}_p - \mathbf{r}|}$, $\mathbf{r} = (x, y, z)$ as a Taylor series and take into account only the most relevant terms.

b) Calculate the probability that the electron makes the transition described in item a) if $|E_m - E_n| \ll \frac{\hbar v}{d}$. You need to evaluate the time integral arising from perturbation theory. *(1.5 points)*
Hint: To calculate the integral, make the change of variables $\frac{vt}{d} = \tan\theta$. Note that the characteristic time during which the particle and the electron interact is of the order of d/v. For $|t| \gg d/v$ the integrand is small.

c) Show that if $\Delta E_{\min} \gg \frac{\hbar v}{d}$, with $\Delta E_{\min}$ the minimum separation between electron bound-state energies, the moving particle does not induce transitions between bound states. Give an interpretation to the quantity $\frac{\hbar v}{d}$. *(1 point)*

Problem 2 *Time-dependent perturbation theory: Symmetries*

A large group of hydrogen atoms in their ground state is between the plates of a parallel-plate capacitor. A voltage pulse is applied to the capacitor so that it produces a homogeneous electric field

$$
\begin{aligned}
E &= 0, &\text{for } t < 0,\\
E(t) &= E_0 e^{-\frac{t}{T}}\mathbf{e}_z, &\text{for } t > 0.
\end{aligned}
$$

a) Show that

$$\int_0^\infty r^{n-1} e^{-\beta r} dr = \frac{(n-1)!}{\beta^n} \qquad (1)$$

for positive integers n. *(0.5 points)*

b) After a long time, what fraction of the atoms have been excited to the $2p$ $(m = 0)$ state? *(1.5 points)*

Hint: Use Eq. (??). The eigenstates corresponding to the quantum numbers ($n = 1$, $l = 0$, $m = 0$) and ($n = 2$, $l = 1$, $m = 0$), respectively, are

$$\psi_{100}(r, \theta, \phi) = \frac{1}{\sqrt{\pi} a_0^{3/2}} e^{-\frac{r}{a_0}}, \quad \psi_{210}(r, \theta, \phi) = \frac{r \cos\theta}{\sqrt{\pi}(2a_0)^{5/2}} e^{-\frac{r}{2a_0}}.$$

Here r is the radial coordinate, θ the polar angle, ϕ the azimuthal angle, and a_0 is the Bohr radius.

c) What fraction of the atoms will be excited to the $2s$ state? It is sufficient to treat the problem to the first order in perturbation theory. *(0.5 points)*

Hint: The eigenstate corresponding to the quantum numbers ($n = 2$, $l = 0$, $m = 0$) is

$$\psi_{200}(r, \theta, \phi) = \frac{1}{4\sqrt{2\pi} a_0^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-\frac{r}{2a_0}}.$$

d) Show that within first order time-dependent perturbation theory the probability that a general time-varying potential $\hat{V}(t)$ induces transitions from an initial state $|j\rangle$ at $t = 0$ to a state $|k\rangle$ at $t = t_0$ equals the probability to induce transitions from $|k\rangle$ at $t = 0$ to $|j\rangle$ at $t = t_0$.

$$P_{jk}(t_0) = P_{kj}(t_0).$$

This result is called *the principle of detailed balance.* Although here proved only up to the first order in perturbation theory, it is generally true. *(1 point)*

Hint: $\hat{V}(t)$ is Hermitian.

e) If $\hat{V}(t)$ and $|k\rangle$, $|j\rangle$ in item d) are the potential and the states considered in item b), respectively, the principle of detailed balance can be applied. What is wrong in the following reasoning: "By the principle of detailed balance the probabilities for the transitions $1s \to 2p$ $(m = 0)$ and $2p$ $(m = 0) \to 1s$ are the same. Therefore the electric field does not change the fraction of atoms in the state $2p$ $(m = 0)$." *(0.5 points)*

Problem 3 *Yukawa potential*

Consider a potential of the form

$$V(\mathbf{r}) = V_0 \frac{e^{-\alpha r}}{r}, \qquad (2)$$

where V_0 and α are real constants and $\alpha > 0$. This potential is called Yukawa potential. In the following, assume that $|V_0|$ is sufficiently small for the Born approximation to be valid.

a) Find the scattering amplitude. *(1 point)*

b) Find the differential scattering cross section and show that it does not depend on the azimuthal angle and that the cross section is larger in the forward direction than in the backward. *(2 point)*

c) In the limit $\alpha \to 0$, the Yukawa potential approaches the Coulomb interaction potential between two charged particles. What is the form of V_0 in this case? What is the differential cross section? *(1 point)*
Note: The expression found above is indeed that of the Coulomb scattering cross section, called Rutherford's formula. While within the Born approximation the Yukawa potential gives precisely the Rutherford's formula, the theory we used to find it is not applicable to the Coulomb potential and therefore does not constitute a proof that the Coulomb cross section follows Rutherford's formula.

Problem 4 *Scattering by a square-well potential*

Consider scattering of particles interacting via a 3D square well potential: $V(r < a) = V_0$; $V(r > a) = 0$.

a) Evaluate the Born approximation scattering cross section for the above potential. *(1 point)*

b) Find the low-energy limit up to second order. *(1 point)*

c) Evaluate the total cross section in the limits of low and high energy. *(2 point)*