# Theoretical Physics V

**SS 2014** **7.5.2014**
**Assignment IV** **Due date 14.5.2014**

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## Problem 1 *Plane wave expansion*

A plane wave can be expressed as a sum of spherical waves

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} a_l j_l(kr) P_l(\cos\theta). \tag{1}$$

Here $j_l(kr)$ is the spherical Bessel function of order $l$ and $P_l$ a Legendre polynomial.

a) Use the orthogonality of the Legendre polynomials

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2m+1}\delta_{mn} \tag{2}$$

to show that

$$\int_0^{\pi} e^{ikr\cos\theta} P_l(\cos\theta)\sin\theta d\theta = \frac{2a_l}{2l+1} j_l(kr). \tag{3}$$

*(1 point)*

b) In the limit $kr \to 0$ the spherical Bessel functions are given by $j_l \approx \frac{2^l l!}{(2l+1)!}(kr)^l$. Use this approximation to show that

$$\int_0^{\pi} (i\cos\theta)^l P_l(\cos\theta)\sin\theta d\theta = \frac{a_l}{2l+1}\frac{2^{l+1} l! l!}{(2l+1)!}. \tag{4}$$

*(1 point)*

c) Use the identity

$$\int_{-1}^{1} x^m P_m(x)dx = \frac{2^{m+1} m! m!}{(2m+1)!} \tag{5}$$

to show that $a_l = i^l(2l+1)$.

*(1 point)*

## Problem 2 *S-wave scattering*

Consider in one dimension particles incident from left that scatter from a delta potential $V(x) = U\delta(x)$.

a) The wave function of a particle can be written in the form

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & \text{if } x < 0, \\ te^{ikx} & \text{if } x > 0, \end{cases} \tag{6}$$

where $r$ and $t$ are the reflection and transmission amplitudes, respectively. The incident wave is described by $e^{ikr}$. (Here the overall normalization of the wave function can be omitted.) Write the boundary conditions for $\psi(x)$ and $\psi'(x)$ at $x = 0$ and solve the reflection and transmission coefficients $R = |r|^2$ and $T = |t|^2$. *(1 point)*
*Hint: To find the boundary condition for $\psi'(x)$ integrate Schrödinger equation over $x$ on a conveniently chosen interval.*

b) Alternatively, the wave function can be cast to the form

$$\psi(x) = e^{ikx} + [f_0(k) + \mathrm{sgn}(x) f_1(k)]e^{ik|x|}, \tag{7}$$

where the first term is the incident wave traveling to the right whereas the second and third term represent the symmetric and antisymmetric parts of the scattered wave. Here $f_0$ and $f_1$ are the amplitudes of the scattered partial waves, and $\mathrm{sgn}(x)$ denotes the sign of the $x$ coordinate. On the other hand, the wave function can be expressed as a sum of even and odd partial waves

$$\psi(x) = A_0 \cos(k|x| + \delta_0) - \mathrm{sgn}(x) A_1 \sin(k|x| + \delta_1). \tag{8}$$

Here $A_0$, $A_1$ are constants and $\delta_0$, $\delta_1$ phase shifts. Compare Eqs. (7) and (8) and derive the following values

$$A_0 = e^{i\delta_0}, \; A_1 = -ie^{i\delta_1}, \; f_0 = \frac{1}{2}\left(e^{2i\delta_0} - 1\right), \; f_1 = \frac{1}{2}\left(e^{2i\delta_1} - 1\right). \tag{9}$$

*(1.5 points)*

c) Total scattering amplitudes on the left and right of the origin are naturally defined by

$$f_+ \equiv f_0 + f_1, \quad f_- \equiv f_0 - f_1. \tag{10}$$

Derive the following values

$$f_+ = f_- = \frac{-1}{1 - ika}, \tag{11}$$

where $a = \frac{\hbar^2}{mU}$ is the scattering length. *(0.5 points)*
*Hint: Use the boundary conditions derived in item a) for $\psi(x)$ expressed in the form of Eq. (7). Note that for $x < 0$ we have $f_- = f_0(k) + \mathrm{sgn}(x) f_1(k)$ and for $x > 0$ we have $f_+ = f_0(k) + \mathrm{sgn}(x) f_1(k)$.*

d) Show that there is only s-wave scattering

$$\delta_1 = 0$$

and that the phase shift is given by

$$\delta_0 = -\arctan\left(\frac{1}{ak}\right).$$

*(1 point)*

*Hint: Use Eqs. (9), (10), and (11).*

e) Calculate the total scattering cross section by using the partial wave expansion $\sigma_{\text{tot}} \equiv |f_-|^2 + |f_+|^2$. The expression for $\sigma_{\text{tot}}$ often contains a factor $4\pi$. Why this is not the case here? *(0.5 points)*

f) Calculate the total scattering cross section by using the optical theorem $\sigma_{\text{tot}} = -2\text{Re}f_+$ and show that the result agrees with the item e. *(0.5 points)*

## Problem 3 *Scattering of the p-wave by a hard sphere*

Consider a central potential of a hard sphere

$$V(r) = 0 \quad for \quad r > r_0 \tag{12}$$
$$= \infty \quad for \quad r < r_0. \tag{13}$$

Let us consider here the $p$-wave scattering ($l = 1$) by this potential.

a) With $R_{k,l}(r) = \frac{1}{r}u_{k,l}(r)$ being the radial component of the scattered partial wave, show that the general solution for $u_{k,1}$ is of the form

$$u_{k,l}(r) = C\left[\frac{\sin(kr)}{kr} - \cos(kr) + a\left(\frac{\cos(kr)}{kr} + \sin(kr)\right)\right], \tag{14}$$

with $C$ and $a$ constants. *(1 point)*

b) If $\delta_1(k)$ is the phase shift acquired during scattering, show that $a = \tan\delta_1(k)$. *(1 point)*

c) Determine $a$ from the boundary condition $u_{k,l}(r_0) = 0$. *(1 point)*

d) Show that, as $k$ approaches zero, $\delta_1(k)$ behaves like $(kr_0)^3$, and is therefore negligible compared to the phase shift from $s$-wave scattering, $\delta_0(k)$. *(1 point)*

## Problem 4 *Spherical potential well*

Consider $s$-wave scattering from a spherical potential-well of depth $U_0$ and radius $R$

$$V(r) = -U_0 \quad for \quad r < R \tag{15}$$
$$= 0 \quad for \quad r > R. \tag{16}$$

a) With a suitable Ansatz, determine the $s$-wave scattering amplitude $f_0(k)$. *(1 point)*

b) What is the phase-shift $\delta_0(k)$? *(1 point)*

c) The $s$-wave scattering length $a$ and effective range $r_e$ can be defined via

$$\cot(\delta_0(k)) = -\frac{1}{ka} + \frac{1}{2}kr_e + O(k^2). \tag{17}$$

Expand $\delta_0(k)$ as a power series in $k$ and show that the scattering length is not bound by the radius $R$, but that all values $-\infty < a < \infty$ are possible. *(2 point)*