Theoretical Physics V

SS 2014 **14.5.2014**
Assignment V **Due date 21.5.2014**

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Problem 1 *The LC transmission line*

The LC oscillator of Fig. 1 is one interesting physical system that can be described by a harmonic oscillator Hamiltonian

$$H = \frac{q^2}{2C} + \frac{\varphi^2}{2L}, \tag{1}$$

where L and C are respectively the inductance and capacitance, q is the charge and φ the magnetic flux conjugate to the charge variable. A transmission line can be regarded as the continuum limit of a linear chain of LC oscillators, as in Fig. 2 where the total classical Hamiltonian is given by

$$\mathcal{H} = \sum_i \left(\frac{q_i^2}{2C} + \frac{\varphi_i^2}{2L} - \frac{1}{L}\varphi_i\varphi_{i+1} \right). \tag{2}$$

a) Find the dispersion relation for the linear chain and determine the first Brillouin zone. *(2 point)*

b) What is the quantum mechanical Hamiltonian describing this system? Find the normal operators associated with it and write the Hamiltonian in its diagonal form. *(2 point)*

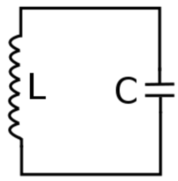

Abbildung 1: LC oscillator

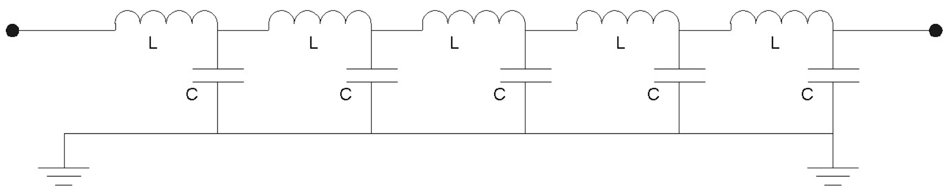

Abbildung 2: Linear chain of LC oscillators.

Problem 2 *Space-time commutation relations*

Since the creation and annihilation operators do not commute, the field operators $\hat{\mathbf{A}}(\mathbf{r}, t)$, $\hat{\mathbf{B}}(\mathbf{r}, t)$ and $\hat{\mathbf{E}}(\mathbf{r}, t)$ will in general not commute.

a) Using the series expansion of the field operators in the Coulomb gauge, evaluate the space-time commutation relations

- $\left[\hat{E}_i(\mathbf{r}_1, t_1), \hat{E}_j(\mathbf{r}_2, t_2)\right]$.

- $\left[\hat{E}_i(\mathbf{r}_1, t_1), \hat{B}_j(\mathbf{r}_2, t_2)\right]$.

Hint: replace the sum over the discrete momentum variables by an integral with respect to continuous variables

$$\sum_{\mathbf{K}} \to \left(\frac{L}{2\pi}\right)^3 \int d^3k \tag{3}$$

and make use of the singular function

$$D(\mathbf{r}, t) \equiv -\frac{1}{(2\pi)^3} \int \frac{d^3k}{k} e^{i\mathbf{k}\cdot\mathbf{r}} \sin(\omega t), \tag{4}$$

whose angular integrals can be easily carried out in spherical coordinates. *(2 point)*

b) From the quantized Hamiltonian for the field, show that the Heisenberg equations of motion for fields $\hat{\mathbf{E}}(\mathbf{r}, t)$ and $\hat{\mathbf{B}}(\mathbf{r}, t)$ coincide with Maxwell equations for the free fields. *(2 point)*

Problem 3 *Kinetic and canonical momenta*

a) The Lagrangian for a particle with mass m and charge q in electromagnetic field is

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{m}{2}\dot{\mathbf{x}}^2 - qU(\mathbf{x}, t) + q\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}, t). \tag{5}$$

Here U and $\mathbf{A}$ are scalar and vector potentials, respectively, that are defined by the equations

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla U - \frac{\partial \mathbf{A}}{\partial t}, \tag{6}$$

where $\mathbf{B}$ is the magnetic field and $\mathbf{E}$ the electric field. Derive the Lorentz equation

$$m\ddot{\mathbf{x}} = q\mathbf{E} + q\dot{\mathbf{x}} \times \mathbf{B} \tag{7}$$

by starting from Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = \frac{\partial L}{\partial x_j}. \tag{8}$$

Above the index j refers to the Cartesian components x, y, and z of the position coordinate.
Hint: Use the vector identity

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + \mathbf{F} \times (\nabla \times \mathbf{G}). \tag{9}$$

(1 point)

b) The quantity $\mathbf{p}^{kin} \equiv m\dot{\mathbf{x}}$ is called *kinetic* momentum while the *canonical* momentum is defined by $p_j^{can} \equiv \frac{\partial L}{\partial \dot{x}_j}$. Calculate $\mathbf{p}^{can}$ for the Lagrangian (5) and express it in terms of $\mathbf{p}^{kin}$. *(0.25 points)*

c) Given the Lagrangian (5), write the classical Hamiltonian $H(\mathbf{x}, \mathbf{p}^{can}) \equiv \mathbf{p}^{can} \cdot \dot{\mathbf{x}} - L$. You need to express H in terms of $\mathbf{x}$ and $\mathbf{p}^{can}$. Show that $\mathbf{p}^{can}$ satisfies Hamilton's equation

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j} \tag{10}$$

and that the Hamilton's equation

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j} \tag{11}$$

implies Lorentz equation (7). Does $\mathbf{p}^{kin}$ satisfy Hamilton's equations? *(0.75 points)*

d) In the quantization of the theory the operators $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}^{can} = -i\hbar\nabla$ for position and canonical momentum are introduced. They satisfy the canonical commutation relation $[\hat{\mathbf{x}}, \hat{\mathbf{p}}^{can}] = i\hbar$. Write the operator $\hat{\mathbf{p}}^{kin}$ describing kinetic momentum in terms of $\hat{\mathbf{p}}^{can}$ and the vector potential $\mathbf{A}$. Show that the expectation value of the kinetic momentum $\langle \hat{\mathbf{p}}^{kin} \rangle$ is invariant under the *gauge transformations*

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\chi, \tag{12}$$

$$U \to U' = U - \frac{\partial \chi}{\partial t}, \tag{13}$$

$$|\psi\rangle \to |\psi'\rangle = e^{iq\chi/\hbar}|\psi\rangle. \tag{14}$$

Here $\chi = \chi(\mathbf{x}, t)$ is an arbitrary scalar function and $|\psi\rangle$ an arbitrary quantum state. Is $\langle \hat{\mathbf{p}}^{can} \rangle$ invariant under gauge transformations? *(0.5 points)*

e) Consider a Lagrangian $L(x_1, x_2, \dot{x}_1, \dot{x}_2)$ that depends on two variables x_1 and x_2 and introduce the complex variable $X = \frac{1}{\sqrt{2}}(x_1 + ix_2)$. Let us show that X and its complex conjugate X^* can be treated formally as independent variables. In particular, show that X and X^* satisfy Lagrange's equations, i.e.,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = \frac{\partial L}{\partial X}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}^*}\right) = \frac{\partial L}{\partial X^*}. \tag{15}$$

(0.75 points)

Hint: Use the chain rule for partial derivatives.

f) For the Lagrangian of item e it is convenient to define canonical momentum by

$$P \equiv \left(\frac{\partial L}{\partial \dot{X}}\right)^*. \tag{16}$$

Prove the following commutation relations

$$[\hat{X}, \hat{P}] = [\hat{X}^\dagger, \hat{P}^\dagger] = 0, \quad [\hat{X}^\dagger, \hat{P}] = [\hat{X}, \hat{P}^\dagger] = i\hbar. \tag{17}$$

Why is the complex conjugation in the definition (16) convenient? *(0.75 points)*
Hint: $\hat{x}_1, \hat{p}_1^{can}$ and $\hat{x}_2, \hat{p}_2^{can}$ satisfy canonical commutation relations.