Theoretical Physics V

SS 2014 **21.5.2014**
Assignment VI **Due date 28.5.2014**

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Problem 1 *Casimir effect*

Consider a large cubic cavity of dimensions $L \times L \times L$ that is bounded by conducting walls. A conducting plate of dimensions $L \times L$ is placed at a distance $\Delta L \ll L$ from one of the faces, so that it divides the cavity in two halves, cf. Fig. 1. By Maxwell's equations and boundary conditions it can be shown that the electric field inside a cavity (either the large cavity without the plate or one of the halves) can be represented as a linear combination of mode functions

$$\mathbf{E}(\mathbf{x}, t) = \sum_n a_n(t)\mathbf{u}_n(\mathbf{x}). \tag{1}$$

The vector components of the mode functions are

$$\begin{aligned} u_x &= C_1 \cos(k_1 x)\sin(k_2 y)\sin(k_3 z), \\ u_y &= C_2 \sin(k_1 x)\cos(k_2 y)\sin(k_3 z), \\ u_z &= C_3 \sin(k_1 x)\sin(k_2 y)\cos(k_3 z), \end{aligned} \tag{2}$$

and the wave vector components are

$$k_1 = \frac{n_1\pi}{L_1}, \quad k_2 = \frac{n_2\pi}{L_2}, \quad k_3 = \frac{n_3\pi}{L_3}, \tag{3}$$

with n_1, n_2, and n_3 nonnegative integers. Moreover, for each (k_1, k_2, k_3) there are two linearly independent polarizations [directions of (C_1, C_2, C_3)], except when one of the components (k_1, k_2, k_3) vanishes, in which case there is only a single mode.
The *zero-point energy* for all the modes in a cavity is given by

$$E_0 = \frac{\hbar}{2}\sum_k \omega_k g_c\left(\frac{k}{k_c}\right). \tag{4}$$

Here ω_k is the mode frequency and the cutoff function g has the properties

$$\begin{aligned} g_c\left(\frac{k}{k_c}\right) &\to 1, \quad k \ll k_c, \\ g_c\left(\frac{k}{k_c}\right) &\to 0, \quad k \gg k_c. \end{aligned} \tag{5}$$

One reason for the existence of the cutoff function is that at sufficiently high frequencies the metal plates do not behave as conductors, and therefore do not affect the electromagnetic field. The detailed form of g_c depends on the specifics of the materials, but will not be needed here.

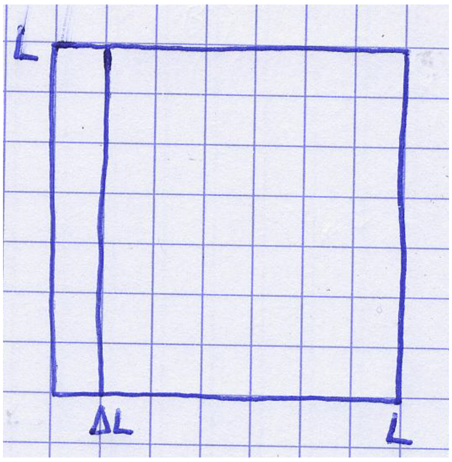

Abbildung 1: A cavity with a plate placed close to one end.

a) Show that the change in the electromagnetic energy within the cavity due to insertion of the plate is

$$\Delta E = \frac{\hbar c L^2}{\pi^2}\left[\sum_{n=0}^{\infty}\left(1-\frac{\delta_{n,0}}{2}\right) f\left(\frac{n\pi}{\Delta L}\right) - \frac{\Delta L}{\pi}\int_0^{\infty} f(k_x)dk_x\right], \quad f(k_x) = \int\int_0^{\infty} g_c\left(\frac{k}{k_c}\right) k dk_y dk_x, \tag{6}$$

with $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$. *(1.5 points)*

Hint: Calculate the zero-point energies for the two halves of the cavity in the presence of the plate and subtract the zero-point energy for the cavity in the absence of the plate. For those linear dimensions of the cavity that are large it is possible to approximate the sum over discrete wave vectors by an integral, cf. Exercise II/1d.

b) Calculate the force per unit area between the conducting plates at $x = 0$ and $x = \Delta L$, $F = -\frac{1}{L^2}\frac{\partial \Delta E}{\partial \Delta L}$. This is called the *Casimir force*. Do the plates at $x = 0$ and $x = \Delta L$ attract or repel each others? Present a mechanical analogue that illustrates Casimir force. *(2.5 points)*

Hint: To simplify f it is convenient to introduce polar coordinates and use the fact that $\int_0^{\infty}\int_0^{\infty}(\ldots)dk_y dk_z = \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\ldots)dk_y dk_z$ when the integrand is invariant under the sign change of k_x, k_y. It is possible to cast ΔE to the form

$$\Delta E = D\left[\sum_{n=0}^{\infty}\left(1-\frac{\delta_{n,0}}{2}\right) G(n) - \int_0^{\infty} G(n)dn\right], \tag{7}$$

where you need to find the constant D and the function G. To simplify (7) you can use the Euler-Maclaurin formula

$$S - \int_m^n G(x)dx = \frac{B_2}{2!}[G'(n) - G'(m)] + \frac{B_4}{4!}[G'''(n) - G'''(m)] + R, \tag{8}$$

where S is the sum $S = \frac{1}{2}f(m) + f(m+1) + \ldots + f(n-1) + \frac{1}{2}f(n)$, the coefficients B_i are given by $B_2 = \frac{1}{6}$ and $B_4 = -\frac{1}{30}$, and the residual term R can be neglected here.

Problem 2 *Correlation functions*

The quantized electric field operator (in the Coulomb gauge) found in class can be separated into the sum of positive and negative frequency parts $\mathbf{E}(\mathbf{r},t) = \mathbf{E}^{(+)}(\mathbf{r},t) + \mathbf{E}^{(-)}(\mathbf{r},t)$, where

$$\mathbf{E}^{(+)}(\mathbf{r},t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}} \mathcal{E}_{\mathbf{k}} a_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)}$$
$$\mathbf{E}^{(-)}(\mathbf{r},t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}} \mathcal{E}_{\mathbf{k}} a^{\dagger}_{\mathbf{k}} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_{\mathbf{k}} t)}.$$

Optical detectors usually use the photoelectric effect to make local field measurements. In such detectors a photon is annihilated and a photocurrent is detected. The probability for the photon to be detected in an interval from t to $t+dt$ is therefore proportional to

$$w_1(\mathbf{r},t) = \sum_f \left| \langle f | \mathbf{E}^{(+)}(\mathbf{r},t) | i \rangle \right|^2 = \langle i | \mathbf{E}^{(-)}(\mathbf{r},t) \mathbf{E}^{(+)}(\mathbf{r},t) | i \rangle ,$$

where the sum is taken over all possible final states $|f\rangle$ and $|i\rangle$ is the initial state of the field (before detection). For a general (non pure) initial state ρ, this reads

$$\begin{aligned} w_1(\mathbf{r},t) &= \mathrm{Tr}\left[\rho \mathbf{E}^{(-)}\mathbf{r},t)\mathbf{E}^{(+)}(\mathbf{r},t)\right] \\ &= \left\langle \mathbf{E}^{(-)}(\mathbf{r},t)\mathbf{E}^{(+)}(\mathbf{r},t) \right\rangle . \end{aligned}$$

This equation can be generalized to define the first-order correlation function

$$G^{(1)}(\mathbf{r}_1,\mathbf{r}_2;t_1,t_2) = \left\langle \mathbf{E}^{(-)}(\mathbf{r}_1,t_1)\mathbf{E}^{(+)}(\mathbf{r}_2,t_2) \right\rangle . \tag{9}$$

If we had considered the probability measurement coming from two detectors we would have found

$$w_2(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2) = \left\langle \mathbf{E}^{(-)}(\mathbf{r}_1,t_1)\mathbf{E}^{(-)}(\mathbf{r}_2,t_2)\mathbf{E}^{(+)}(\mathbf{r}_2,t_2)\mathbf{E}^{(+)}(\mathbf{r}_1,t_1) \right\rangle ,$$

from which we can define the second-order correlation function

$$G^{(2)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4;t_1,t_2,t_3,t_4) = \left\langle \mathbf{E}^{(-)}(\mathbf{r}_1,t_1)\mathbf{E}^{(-)}(\mathbf{r}_2,t_2)\mathbf{E}^{(+)}(\mathbf{r}_3,t_3)\mathbf{E}^{(+)}(\mathbf{r}_4,t_4) \right\rangle . \tag{10}$$

In the exercises below, consider the special case in which the field only has components with two wavevectors $\mathbf{k}$ and $\mathbf{k}'$ with the same polarization ($\epsilon_{\mathbf{k}} = \epsilon_{\mathbf{k}'}$) and frequencies.

a) *Michelson interferometer:* Consider a Young double-slit experiment where the light field at the measuring screen is generated from two point-like sources at positions $\mathbf{r}_1$ and $\mathbf{r}_2$.

- Show that

$$\begin{aligned} G^{(1)}(\mathbf{r},\mathbf{r},t,t) =& G^{(1)}(\mathbf{r}_1,\mathbf{r}_1,t-t_1,t-t_1) + G^{(1)}(\mathbf{r}_2,\mathbf{r}_2,t-t_2,t-t_2) \\ &+ G^{(1)}(\mathbf{r}_1,\mathbf{r}_2,t-t_1,t-t_2) + G^{(1)}(\mathbf{r}_2,\mathbf{r}_1,t-t_2,t-t_1), \end{aligned}$$

  where $t_1(t_2)$ is the time needed for the light from the source located at $\mathbf{r}_1(\mathbf{r}_2)$ to travel to $\mathbf{r}$. *(1 point)*

- Show that $G^{(1)}(\mathbf{r}_1, \mathbf{r}_2, t, t) + G^{(1)}(\mathbf{r}_2, \mathbf{r}_1, t, t)$ is proportional to $\cos\left[(\mathbf{k}+\mathbf{k}')(\mathbf{r}_1 - \mathbf{r}_2)/2\right]$. *(1 point)*

b) *Hanbury-Brown – Twiss experiment:* Determine the second-order correlation functions $G^{(2)}(\mathbf{r}_1, \mathbf{r}_2; t, t) \equiv G^{(2)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_2, \mathbf{r}_1; t, t, t, t)$ for the states given below:

- A two photon state $|1_\mathbf{k}, 1_{\mathbf{k}'}\rangle$, where the field has one photon in mode $\mathbf{k}$ and one photon in mode $\mathbf{k}'$. *(2 point)*
- The light from a star, *i.e.*, the field of a thermal state, where $\langle n^2 \rangle = 2\langle n \rangle^2 + \langle n \rangle$ and $\langle n \rangle = \left[\exp\left(\hbar\omega/k_B T\right) - 1\right]^{-1}$ and n is the number operator. Assume $\langle n_\mathbf{k} \rangle = \langle n_{\mathbf{k}'} \rangle = \langle n \rangle$ and $\langle n_\mathbf{k}^2 \rangle = \langle n_{\mathbf{k}'}^2 \rangle = \langle n^2 \rangle$. *(2 point)*
- A laser field for which $\langle n^2 \rangle = \langle n \rangle^2 + \langle n \rangle$. Assume $\langle n_\mathbf{k} \rangle = \langle n_{\mathbf{k}'} \rangle = \langle n \rangle$ and $\langle n_\mathbf{k}^2 \rangle = \langle n_{\mathbf{k}'}^2 \rangle = \langle n^2 \rangle$. *(2 point)*