Theoretical Physics V

SS 2014 **28.5.2014**
Assignment VII **Due date 4.6.2014**

Prof. Dr. Wilhelm-Mauch http://qsolid.uni-saarland.de/?Lehre:TP_V

Problem 1 *Induced emission and absorption*

Consider a hydrogen atom that at time $t = 0$ is in the eigenstate $|i\rangle$ (with the energy $\hbar\omega_i$) of the atomic Hamiltonian. At time $t = 0$ the atom is subjected to electromagnetic radiation, and at time $t = T$ the radiation is switched off.

a) Suppose the radiation is monochromatic laser light with frequency ω that induces a perturbation described by the Hamiltonian

$$H'(t) = \begin{cases} -q\mathbf{x}\cdot\mathbf{E}, \quad \mathbf{E} = \mathbf{E}_0^* e^{i\omega t} + \mathbf{E}_0 e^{-i\omega t}, \quad \text{if } 0 < t < T \\ 0, \quad \text{if} \quad t < 0 \;\text{ or }\; t > T. \end{cases} \tag{1}$$

Calculate the probability amplitude $c_{i\to f}(T)$ for the transition to the state $|f\rangle$ at time T. Suppose the frequency of the radiation is near resonance $\omega \approx \omega_f - \omega_i > 0$. You can assume that the electric dipole moment of the transition $\boldsymbol{\mu}_{fi} \equiv q\langle f|\mathbf{x}|i\rangle$ and $\mathbf{E}_0$ are given. *(0.75 points)*

Hint: Use first-order perturbation theory.

b) Starting from the result of item (a), calculate the rate of transitions $\Gamma_{abs} \equiv \frac{|c_{i\to f}(t)|^2}{t}$ to the state $|f\rangle$ after a long time $t = T \to \infty$. *(0.75 points)*
Hint: One representation of the Dirac delta function is

$$\frac{2\sin^2\left(\frac{1}{2}aT\right)}{\pi a^2 T} \overset{T\to\infty}{\to} \delta(a).$$

c) Solve the problem of item (b) by starting directly from Fermi's golden rule. Show that the result coincides with the item (b). *(0.25 points)*

d) Natural light is usually *incoherent* and *unpolarized*. This means that the phase of the light emitted by the light source varies randomly and all directions for the electric field are equally probable. Suppose now that instead of laser light the atom is subjected to incoherent unpolarized light. Show that the transition probability to the state $|f\rangle$ after time T is

$$|c_{i\to f}(T)|^2 = \frac{1}{3}\left\{\frac{2\sin\left[\frac{1}{2}(\omega_{fi}-\omega)T\right]|\mathbf{E}_0||\boldsymbol{\mu}_{fi}|}{\hbar(\omega_{fi}-\omega)}\right\}^2, \quad \omega_{fi} \equiv \omega_f - \omega_i.$$

(0.5 points)

e) Suppose the atom is subjected to incoherent unpolarized radiation whose *time averaged* energy density (energy per unit volume) in a narrow frequency range $\Delta\omega$ is $u(\omega)\Delta\omega$. Calculate $\Gamma_{abs} \equiv \frac{|c_{i\to f}(T)|^2}{T}$ after a long time $T \to \infty$.
Hint: The instantaneous energy of the radiation field enclosed in the volume V is $W = \int_V dv\varepsilon_0 |\mathbf{E}|^2$. You have to express $|\mathbf{E}_0|^2$ in terms of $u(\omega)\Delta\omega$. In this part you can assume that $u(\omega)$, but not $\mathbf{E}_0$, is given. *(1.25 points)*

f) For the setup of item (e), calculate the rate for stimulated emission Γ_{se}, *i.e.*, the transition probability per unit time for the transition from the state $|f\rangle$ to the state $|i\rangle$. There is a simple relation between Γ_{abs} and Γ_{se}. What is this phenomenon called? *(0.5 points)*

Problem 2 *Spontaneous emission*

a) Consider a hydrogen atom that is coupled to electromagnetic field through the Hamiltonian

$$\hat{H}' = -\frac{e}{mc}\hat{\mathbf{A}} \cdot \hat{\mathbf{p}}, \tag{2}$$

where the quantized vector potential is

$$\hat{\mathbf{A}}(\mathbf{x}) = \sum_\alpha \sum_k \sqrt{\frac{c^2\hbar}{2\omega V}} \left(\epsilon_{\mathbf{k},\alpha} e^{i\mathbf{k}\cdot\mathbf{r}} \hat{a}_k(t) + \epsilon_{\mathbf{k},\alpha} e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{a}_k^\dagger(t) \right).$$

Here $\epsilon_{\mathbf{k},\alpha}$ $(\alpha = 1, 2)$ are two linearly independent unit vectors describing polarization and satisfying $\mathbf{k} \cdot \epsilon_{\mathbf{k},\alpha} = 0$. At time $t = 0$ the atom is in its exited state $|e\rangle$ and there is no incident electromagnetic wave. Let us study the probability that the the atom makes spontaneously a transition to the ground state through photon emission. Using Fermi's golden rule, write the transition rate Γ_{sp} (transition probability per unit time) from the state $|e, n_i\rangle$ to the state $|g, n_f\rangle$. Here $n_i = 0$ and $n_f = 1$ are the photon numbers in the initial and final state, respectively. The photon density of states per unit solid angle is $\frac{d\rho(\hbar\omega)}{d\Omega} = \frac{V\omega^2}{(2\pi)^2\hbar c^3}$. In this part you only need to express Fermi's golden rule through the given quantities but not to evaluate the integrals. *(0.5 points)*

b) Show that the rate for spontaneous emission with the photon emission to the solid angle $d\Omega_{\mathbf{k}}$ is

$$\frac{d\Gamma_{sp}}{d\Omega_{\mathbf{k}}} = \frac{e^2\omega}{8\pi^2 m^2 \hbar c^3} \sum_\alpha |\langle g|\hat{\mathbf{p}}|e\rangle \cdot \epsilon_{\mathbf{k},\alpha}|^2 d\Omega_{\mathbf{k}} \tag{3}$$

(1 point)
Hint: In a typical atomic transition in the optical region the wave length of the emitted photon is much larger than the size of the atom. You can assume this is the case here.

c) For the atomic Hamiltonian $\hat{H}_0 = \frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{x})$ let us denote the the eigenenergies corresponding to eigenstates $|g\rangle$ and $|e\rangle$ by $\hbar\omega_g$ and $\hbar\omega_e$, respectively. Express the matrix element $\langle g|\hat{\mathbf{p}}|e\rangle$ in terms of the electric dipole moment $\boldsymbol{\mu}_{ge} \equiv -e\langle g|\mathbf{x}|e\rangle$. *(1 point)*
Hint: Use canonical commutation relations.

d) Calculate Γ_{sp} in terms of $\boldsymbol{\mu}_{ge}$. *(1 point)*
Hint: Use the result of item (c).

e) What is wrong in the following reasoning? "For all photon numbers n the expectation value $\langle n|\hat{\mathbf{A}}|n\rangle = 0$ therefore $\hat{\mathbf{A}}$ does not have physical consequences." *(0.5 points)*

Problem 3 *Light scattering*

In perturbation theory we divide our Hamiltonian into a large term (which we know how to threat) and a small perturbation $H = H_0 + V$. The formal solution of the time-evolution operator in the interaction picture, is given by

$$U^I(t_1, t_2) = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_{t_2}^{t_1} dt' V^I(t')\right), \tag{4}$$

where $\mathcal{T}$ is the Dyson time-ordering operator. Using the Dyson series expansion of U^I, let us see what can we learn about light scattering from an atom, *i.e.* V is the dipole Hamiltonian and we want to calculate transitions amplitudes from $|i, 1_k\rangle$ to $|f, 1_l\rangle$, where i and f are arbitrary initial and final atomic states and $|1_k\rangle$ ($|1_l\rangle$) represents a one excitation Fock state in mode k (l).

a) Show that to zero order $\langle f, 1_l| U_0^I |i, 1_k\rangle = \delta_{i,f}\delta_{k,l}$. We also no know to first order no scattering will occur, the photon in absorbed and emitted into the same mode. *(1 point)*

b) Without explicitly evaluating the time integrals, show that in second order perturbation, the scattering amplitude separates into two terms with distinct intermediate photonic states. Interpret these two processes. *(2 point)*

c) Evaluate the previous integrals and show that the time dependence of both processes is the same. *(2 point)*

d) Draw Feynmann diagrams corresponding to these scattering processes. *(1 point)*