Theoretical Physics V

SS 2014 **11.6.2014**
Assignment IX **Due date 18.6.2014**

Prof. Dr. Wilhelm-Mauch http://qsolid.uni-saarland.de/?Lehre:TP_V

Problem 1 *Superconducting correlations*

The Nobel Prize in physics in 1972 was awarded to J. Bardeen, L. N. Cooper and J. R. Schrieffer for their theory of superconductivity, usually called the BCS-theory. Accordingly the ground state discussed in Exercise XIII/1

$$|\Psi\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} a^{\dagger}_{\mathbf{k},\uparrow} a^{\dagger}_{-\mathbf{k},\downarrow})|0\rangle \tag{1}$$

is also called the BCS ground state. The BCS ground state $|\Psi\rangle$ is not an eigenstate of the number operator but can be expanded in the form

$$|\Psi\rangle = \sum_{N} c_N |\Psi_N\rangle \tag{2}$$

where $|\Psi_N\rangle$ is the normalized component of $|\Psi\rangle$ with N electrons (creation operators acting on $|0\rangle$). The coefficients satisfy $\sum_N |c_N|^2 = 1$ to assure the normalization of $|\Psi\rangle$. In expansion (2), c_Ns vanish for odd N. The magnitude of even c_Ns is peaked around the expectation value of the number operator N_0 and is negligible when the distance $|N - N_0|$ is considerably larger than $\sqrt{N}$. On a scale $|N - N_0| \ll \sqrt{N}_0$ the even coefficients c_N vary only a little.

a) Let N be close to the expectation value of the number operator in the state $|\Psi\rangle$. Show that if an operator $\hat{A}$ acting on $|\Psi_N\rangle$ gives a state with $N + p$ particles with $p \ll \sqrt{N}$ and if the matrix element $\langle\Psi_{N+p}|\hat{A}|\Psi_N\rangle$ varies slowly with N on the scale $\sqrt{N}$ then

$$\langle\Psi_{N+p}|\hat{A}|\Psi_N\rangle = \langle\Psi|\hat{A}|\Psi\rangle. \tag{3}$$

(1.75 points)

b) Suppose two electrons are added to the states $\mathbf{k}\uparrow$ and $-\mathbf{k}\downarrow$ in the state $|\Psi_N\rangle$. The probability amplitude of obtaining the state $|\Psi_{N+2}\rangle$

$$F_{\mathbf{k}} \equiv \langle\Psi_{N+2}|a^{\dagger}_{\mathbf{k}\uparrow} a^{\dagger}_{-\mathbf{k}\downarrow}|\Psi_N\rangle \tag{4}$$

is called the *condensation amplitude* or *pairing amplitude*. Calculate $F_{\mathbf{k}}$ in terms of coherence factors $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$. *(1.25 points)*
Hint: Use Eq. (3).

c) Calculate the expectation value $\langle\tilde{\Psi}|a^{\dagger}_{\mathbf{k}\uparrow} a^{\dagger}_{-\mathbf{k}\downarrow}|\tilde{\Psi}\rangle$ when $|\tilde{\Psi}\rangle$ describes noninteracting Fermi gas in its ground state

$$|\tilde{\Psi}\rangle = \prod_{k<k_F} a^{\dagger}_{\mathbf{k},\uparrow} a^{\dagger}_{-\mathbf{k},\downarrow}|0\rangle. \tag{5}$$

Here k_F is the magnitude of the Fermi wave vector. Compare the result with $\langle\Psi|a^{\dagger}_{\mathbf{k}\uparrow} a^{\dagger}_{-\mathbf{k}\downarrow}|\Psi\rangle$ and interpret the result. *(1 point)*

d) Calculate $\langle\Psi|a^\dagger_{\mathbf{k}\sigma}a^\dagger_{\mathbf{k}'\sigma'}|\Psi\rangle$ when $(\mathbf{k}',\sigma')\neq(-\mathbf{k},-\sigma)$. *(1 point)*

e) Calculate $\langle\Psi|a^\dagger_{\mathbf{k}\sigma}a_{\mathbf{k}'\sigma'}|\Psi\rangle$ when $(\mathbf{k}',\sigma')\neq(\mathbf{k},\sigma)$. *(1.25 points)*

f) In items (b),(d), and (e) an expectation value was calculated for three different operators (correlators). Suppose that in an *a priori* unknown state, the expectation value of each operator vanishes. Is the state a BCS type superconducting state? *(0.25 points)*

Problem 2 *The Jordan-Wigner Transformation*

Many quantum systems are well-described as 'chains' of two-level systems, each of which has the same Hilbert space as a spin-1/2 particle. Valid Hamiltonians acting on a single two-level system linear combinations of the traceless, Hermitian Pauli matrices and the two-state identity:

$$\sigma^x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma^z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \text{and}\ \mathbb{1}^2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

From these, one can construct the raising-operator $\sigma^+ \equiv \sigma$ and the lowering-operator $\sigma^- = \sigma^\dagger$:

$$\begin{aligned} \sigma &= \frac{\sigma^x + i\sigma^y}{2} \\ \sigma^\dagger &= \frac{\sigma^x - i\sigma^y}{2}. \end{aligned} \tag{7}$$

a) Show that the σ and $\sigma^\dagger$ obey single-particle fermionic anticommutation relations:

$$\{\sigma,\sigma^\dagger\} = \mathbb{1}^2 \tag{8}$$

$$\{\sigma,\sigma\} = \{\sigma^\dagger,\sigma^\dagger\} = 0 \tag{9}$$

(1 point)

b) Hamiltonians describing k coupled two-level systems can be expressed as linear combinations of these single-spin operators, each addressing a single site. In the full Hilbert space, σ acting on the jth site, σ_j, is expressed mathematically as:

$$\begin{aligned} \sigma_j &\equiv \mathbb{1}^2 \otimes \mathbb{1}^2 \otimes ... \otimes \mathbb{1}^2 \otimes \sigma \otimes \mathbb{1}^2 \otimes .... \otimes \mathbb{1}^2 \\ &= (\otimes_{i=1}^{j-1}\mathbb{1}^2) \otimes \sigma \otimes (\otimes_{i'=j+1}^{k}\mathbb{1}^2) \end{aligned} \tag{10}$$

Practice working with this notation by showing that

$$\{\sigma_i, \sigma_j^\dagger\} \neq 0 \tag{11}$$

for $i \neq j$. This means that the $\{\sigma_i\}$ are not fermionic creation operators for more than one particle.
(1 point)

c) The Jordan-Wigner transformation tells us how to map a 'spin chain', an array of two level systems, to a system of more than one fermion. This transformation allows us to study elementary fermions on a spin chain, which can be engineered, for example, from the quantum bits (qubits) of a quantum computer.

The Jordan-Wigner transformation is a mapping from $\{\sigma_j\}$ and $\{\sigma_j^\dagger\}$ to another set of operators, $\{a_j\}$ and $\{a_j^\dagger\}$, that do in fact obey *all* the fermionic anticommutation relations.

Show that the annihilation operators

$$a_j \equiv -(\Pi_{i=1}^{j-1}\sigma_i^z)\sigma_j \tag{12}$$

and the corresponding $\{a_j^\dagger\}$ obey fermionic commutation relations,

$$\{a_i, a_j\} = 0 \text{ and } \{a_i, a_j^\dagger\} = \mathbb{1}^{2^k}\delta_{ij},\ \forall\, i, j \tag{13}$$

Hint: $(\sigma_i^z)^2 = (\sigma_i^x)^2 = (\sigma_i^y)^2 = \mathbb{1}^2$, *and* $\sigma_i^z = (\sigma_i^z)^\dagger$, *etc. Prove separately the cases for* $i \neq j$ *and* $i = j$. *You can assume without loss of generality that* $i < j$ *for the cases when* $i \neq j$.
(1 point)

d) Show that the transformation is invertible, and in particular, that

$$\sigma_j^z = a_j a_j^\dagger - a_j^\dagger a_j \tag{14}$$
$$\sigma_j^x = -(\Pi_{l=1}^{j-1}\sigma_l^z)(a_j + a_j^\dagger) \tag{15}$$
$$\sigma_j^y = i(\Pi_{l=1}^{j-1}\sigma_l^z)(a_j - a_j^\dagger) \tag{16}$$

(1 point)