Introduction to quantum information

WS 2012/13 **04.11.2012**
Assignment I **Due date 16.11.2012**

Prof. Dr. Wilhelm-Mauch http://qsolid.uni-saarland.de/?Lehre

Problem 1 *Bipartite systems*

a) Consider a bitartite quantum system consisting of a qubit with basis states $|\mathrm{Q}\rangle \in \{|0\rangle, |1\rangle\}$ and a harmonic oscillator states with an infinite basis of Fock states $|\mathrm{HO}\rangle \in \{|0\rangle, |1\rangle, |2\rangle, ...\}$. The bipartite quantum system is spanned by basis states $|\mathrm{Q}\rangle \otimes |\mathrm{HO}\rangle \equiv |\mathrm{Q}, \mathrm{HO}\rangle$.

What are the maximum and minimum entropy and purity a state in this system can reach? *(1 point)*

b) Consider the so-called Schrödinger cat state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0, \alpha\rangle + |1, -\alpha\rangle)$ where $|\pm\alpha\rangle$ are coherent states of the oscillator, $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^{n/2}}{\sqrt{n!}} |n\rangle$.

Show that, in general, $|\alpha\rangle$ and $|-\alpha\rangle$ are not orthogonal by calculating $\langle\alpha|-\alpha\rangle$. Discuss the relationship between $|\alpha\rangle$ and $|-\alpha\rangle$ in the limit that $|\alpha| \to 0$ and $|\alpha| \to \infty$. *(1 point)*

c) Calculate the reduced density matrices $\rho_{\mathrm{HO}} \equiv \mathrm{tr}_{\mathrm{Q}}\left[|\psi\rangle\langle\psi|\right]$ and $\rho_{\mathrm{Q}} \equiv \mathrm{tr}_{\mathrm{HO}}\left[|\psi\rangle\langle\psi|\right]$. *(1 point)*

d) Show that $|\psi\rangle$ is an entangled state by calculating the purity of ρ_{HO} and ρ_{Q}. How do these purities relate to each other, and to the original purity of $|\psi\rangle$? *(1 point)*

Problem 2 *Mixed states*

a) Consider the density matrix ρ_{Q} of a qubit. Show that there is a pure state $|\psi\rangle$ in a system consisting of the qubit and an ancilla qubit A such that $\rho_{\mathrm{Q}} = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$. *(1 point)*

b) Give a $|\psi\rangle$ for which $\rho_{\mathrm{Q}} = \frac{\hat{1}}{2}$. *(1 point)*

c) Prove the generalization of a) to a d-dimensional system. Show that the dimension of the Ancilla is at most d. *(2 point)*

Problem 3 *Euler angles*

a) Show that, up to a meaningless global phase, any unitary operation on a single qubit can be written as $\hat{R}_z(\gamma)\hat{R}_x(\beta)\hat{R}_z(\alpha)$ where $\hat{R}_i(\phi) = \exp(-i\phi\hat{\sigma}_i)$ represents a rotation around the i-axis by angle ϕ. *(3 points)*

b) Explicitly write out this decomposition for the Hadamard gate and for

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & i\sqrt{2} \\ i\sqrt{2} & 1 \end{pmatrix}.$$

(2 point)

Problem 4 *Purity and entanglement*

a) Show that the Liouville-von-Neuman equation

$$i\hbar\dot{\rho} = [\hat{H}, \rho]$$

conserves the purity of ρ. *(1 point)*

b) We have seen in class that the transpose of a density operator is a positve, but not completely positive operation. Show that the partially transposed density matrix of a two-qubit system has a negative eigenvalue if and only if the two-qubit (pure) state is entangled. *(2 points)*