Introduction to quantum information

WS 2012/13 **31.11.2012**
Assignment 4 **Due date 12.12.2012**

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Problem 1 *Quantum Oracles*

In the Deutsch-Josza algorithm, a quantum oracle transforms the state $|x\rangle \otimes |y\rangle$ to the state $|x\rangle \otimes |f(x) \oplus y\rangle$ where $|x\rangle$ is the state of an n-qubit quantum register, $|y\rangle$ is the state of a single ancillary qubit, and $\oplus$ indicates binary addition. The oracle performs a unitary transformation U_{oracle} such that

$$U_{\text{oracle}} \left(|x\rangle \otimes |y\rangle\right) = |x\rangle \otimes |f(x) \oplus y\rangle. \tag{1}$$

a) Give an explicit circuit diagram for U_{oracle} in the case that $f(x) = 1$.
Hint: $f(x)$ does not depend on x. *(1 point)*

b) Give an explicit circuit diagram for U_{oracle} in the case that $f(x) = 1 - x$.
Hint: Only one digit of x is needed to determine $f(x)$. *(2 point)*

Problem 2 *Superdense coding*

a) Consider a superdense coding protocol where Alice shares a Bell state $|\beta_{ij}\rangle$ with Bob, on which she performs a local operation $\hat{A} \in \{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$. Show that this results in another Bell state, $|\beta_{i'j'}\rangle = \hat{A}\left(|\beta_{ij}\rangle\right)$, uniquely determined by her choice of $\hat{A}$. *(2 points)*

b) Suppose Eve intercepts Alice's qubit (and not Bob's). She can then measure the qubit with respect to any of the local operators $\hat{E} \in \{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$, the result of which is expected value $\langle \beta_{i'j'} | \hat{E} \otimes \mathbb{I}_{\text{B}} | \beta_{i'j'} \rangle$ where $\mathbb{I}_{\text{B}}$ is the identity acting on Bob's qubit. Show that by doing so, Eve cannot resolve which Bell state Alice is sending to Bob. Can she infer anything about which of the four possible strings Alice wants to send? *(2 points)*

Problem 3 *Bell's inequality*

Bell's inequality is a test for local hidden variable theories, i.e., theories that assume that the probabilistic outcome of quantum measurements is due to a hidden variable, a physical variable we do not have experimental access to, which is local.

a) [1]Suppose Alice and Bob have a qubit each, and Alice can measure Q and R on her qubit while Bob can measure S and T on his qubit. The eigenvalues of Q, R, S, and T are $q, r, s, t \in \{1, -1\}$, respectively. A local hidden variable theory would assign a probability density $p(q, r, s, t)$ to the outcome of Alice and Bob measuring q, r, s, and t. Using such a probability density, show that

$$\langle QS + RS + RT - QT \rangle \leq 2,$$

which is known as Bell's inequality. *(2 points)*

b) Consider specifically $\hat{Q} = \hat{Z}_1$, $\hat{R} = \hat{X}_1$, $\hat{S} = -(\hat{Z}_2 + \hat{X}_2)/\sqrt{2}$, $\hat{T} = (\hat{Z}_2 - \hat{X}_2)/\sqrt{2}$ and the qubits in the Bell state $|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. Compute the above expectation value with respect to quantum operators $\hat{Q}$, $\hat{R}$, $\hat{S}$, and $\hat{T}$, showing that this leads to $2\sqrt{2}$, a direct violation of Bell's inequality. *(2 points)*

[1]Part a) of this problem is extra credit.