Introduction to quantum information processing
Exercise sheet 1

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Rules: *You usually hand back the assignments in the Thursday class or before it in the mailbox of AG Wilhelm-Mauch in the entrance area of E2.6. Please provide your name to your solutions. Please hand in in groups of at least three students. You are supposed to be able to present your solutions in the recitation session (Übungsgruppe), if they were handed in in your name. Should you not be able to present that solution, we will interpret this as cheating and you will receive zero points on the complete assignment. The hand-in problems will be marked, and reaching 50% of the points is sufficient to qualify for the exam. The analytic problems of an assignment sheet have 40 points in total. The coding problems are marked in an honor system, i.e., you self-assess what you have done in the beginning of the recitation session. The points you earn with the coding problems count as bonus points.*

Exercise 1: Axioms of quantum mechanics and measurement (*14 points*)

According to the axioms of quantum mechanics

1. each observable A (i.e. a physical quantity like linear or orbital momentum, energy etc.) is represented by an operator $\hat{A}$,
2. the outcome of a measurement of such an observable can only be one of the corresponding operator's eigenvalues
3. and furthermore, having measured the eigenvalue a_i, the state vector $|\psi\rangle$ describing the physical system immediately *collapses* into the corresponding eigenspace of $\hat{A}$ to the eigenvalue a_i:

$$|\psi_{\text{post}}\rangle = \frac{1}{\sqrt{p_i}} \hat{\Pi}_i |\psi\rangle \,, \tag{1}$$

where $\hat{\Pi}_i$ denotes the projection operator onto the corresponding eigenspace.

(a) (i) Give the projection operator $\hat{\Pi}_i$ for the non-degenerate and the degenerate case, (*1 point*)

(ii) show that they are indeed projection operators, i.e. $\hat{\Pi}_i^2 = \hat{\Pi}_i$, (*1 point*)

(iii) show that the post-measurement state vector $|\psi_{\text{post}}\rangle$ given in Eq. (1) is normalised, i.e. $\langle\psi_{\text{post}}|\psi_{\text{post}}\rangle = 1$, (*1 point*)

(iv) and express $|\psi_{\text{post}}\rangle$ in terms of the corresponding eigenvectors $|a_i\rangle$ ($|a_i^{(k)}\rangle$ for the degenerate case). (*1 point*)

(b) As all physical observables are real, the measurement outcome and thus, according to the axioms above, the eigenvalues of the corresponding operators must also be real. Show that an operator $\hat{A}$ that is hermitian, i.e. $\hat{A}^\dagger = \hat{A}$, has only real eigenvalues, and that eigenvectors $|a_i\rangle, |a_j\rangle$ to different eigenvalues $a_i \neq a_j$ are orthogonal, $\langle a_i|a_j\rangle = 0$. This shows, why all physical observables are represented by *hermitian* operators. (*3 points*)

Given are two observables A, B with corresponding operators $\hat{A}, \hat{B}$ and eigenstates $\hat{A}|a_i\rangle = a_i|a_i\rangle, \hat{B}|b_i\rangle = b_i|b_i\rangle$ $(i = 1, 2)$. Furthermore, the states are connected via

$$|a_1\rangle = N_1 (3|b_1\rangle + 4|b_2\rangle)\,, \; |a_2\rangle = N_2 (4|b_1\rangle - 3|b_2\rangle)\,.$$

(c) Determine the normalization constants N_1 and N_2 and show that $|a_1\rangle$ and $|a_2\rangle$ are orthogonal. *(2 points)*

(d) The measurement of observable A gives the result a_1. Observable B is measured immediately (you do not have to take into account any evolution of the state). Give the probability with which this measurement will give a value of b_1 and b_2. *(2 points)*

(e) The observable A is measured once again. Calculate the probability for measuring the value a_2. *(2 points)*

(f) Do $\hat{A}$ and $\hat{B}$ commute, i.e. $\left[\hat{A}, \hat{B}\right] := \hat{A}\hat{B} - \hat{B}\hat{A} = 0$? *(1 point)*

Exercise 2: Bloch sphere *(10 points)*

The purpose of this exercise is to convince ourselves that the Bloch sphere is a visualization of a qubit's two-dimensional Hilbert space such that each state $|\psi\rangle$ is *unambiguously* (up to the irrelevant global phase) represented by a point (x_1, x_2, x_3) on the *unit sphere*, where the coordinates are given by the expectation values $x_i = \langle\sigma_i\rangle = \vec{\psi}^\dagger \cdot \sigma_i \cdot \vec{\psi}$ $(i = 1, 2, 3)$, where the state vector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is represented by a column vector $\vec{\psi} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. This is done in two steps:

(a) Show that $\langle\sigma_1\rangle^2 + \langle\sigma_2\rangle^2 + \langle\sigma_3\rangle^2 = 1$, where the expectation value is taken for an arbitrary state vector $|\psi\rangle$. *(4 points)*

(b) The density operator $\hat{\rho}$ of a system described by the state vector $|\psi\rangle$ is defined as $\hat{\rho} = |\psi\rangle\langle\psi|$, and it can be shown that the expectation value of observable A is given by $\langle A\rangle = \mathrm{Tr}\left\{\hat{\rho}\hat{A}\right\}$. Choosing a basis, this operator is represented by a 2×2-matrix. Show that, in the usual computational basis, this representation is given by

$$\rho = \frac{1}{2}\left(\sigma_0 + \sum_{i=1}^{3} s_i\sigma_i\right), \tag{2}$$

where the coefficients s_i are given by $s_i = \langle\sigma_i\rangle = \mathrm{Tr}\{\sigma_i\hat{\rho}\}$. This shows, that the density matrix - and thus the state of a system - is fully described by the *expectation* values of the Pauli operators. *(3 points)*
Hint: You can start from a general representation Eq. (2) with unknwown coefficients s_i which you can solve for by exploiting the fact that the trace functions similarly to the dot product of vectors.

(c) Using (2), argue that each state vector $|\psi\rangle$ (up to a global phase) can be represented by exactly one point on the Bloch sphere, and vice versa. *(3 points)*

Exercise 3: Entanglement *(16 points)*

In what follows we consider two qubits named A and B and describe their common state in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. In the notation $|\sigma_1\sigma_2\rangle$ $(\sigma_i \in \{0, 1\})$ the first digit σ_1 gives the state of qubit A and the second digit gives the state of qubit B, i.e. $|01\rangle$ means that qubit A is in state $|0\rangle$ and qubit B is in state $|1\rangle$.

(a) Which of the following two-qubit states describe entangled states and thus cannot be written as a product state, and which describe non-entangled states? Give the product state in the latter case.

(i) $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ *(0.5 points)*

(ii) $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$ *(0.5 points)*

(iii) $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. *(0.5 points)*

(b) Consider the operator $\hat{Z}_{AB} := Z_A Z_B$, where Z_i onlfor By acts on qubit i and leaves the other one unchanged (acts as unity on its subspace). What are the eigenvalues and corresponding eigenstates of Z_{AB}? Give the corresponding projection operators $\hat{\Pi}_i$. Express the eigenstates and the projection operators in the computational basis. *(4 points)*

Hint: Technically, Z_{AB} is the tensor product of the two Pauli matrices $Z_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which gives a 4×4-matrix. From that you can determine its eigenvalues and eigenstates - but you could also guess the eigenstates and show that your guess is indeed an eigenstate.

(c) Suppose the two qubits A, B are prepared in the state $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. What are the possible post-measurement states $|\psi_{\text{post}}\rangle$ after measuring $\hat{Z}_{AB}$ and how is this specific class of states called? *(1.5 points)*

Up to here we just dealt with checking whether a two-qubit state is entangled or not. Now, we also want to *quantify* the entanglement, i.e. to say "how much" or "how strong" the two qubits are entangled. There is a bunch of different measures concerning this task, and we are specifically interested in the so called *negativity* measure $\mathcal{N}$. Before giving its definition we must introduce the concept of the *partial transpose*: As the joint state can be written in the computational basis as

$$|\psi\rangle = \sum_{\substack{\sigma_1, \sigma_2 \\ \in \{0,1\}}} c_{\sigma_1,\sigma_2} |\sigma_1 \sigma_2\rangle$$

with coefficients $c_{ij} \in \mathbb{C}$, the same can be done for the density operator,

$$\hat{\rho} = \sum_{\substack{\sigma_1, \sigma_2 \\ \sigma_1', \sigma_2' \\ \in \{0,1\}}} c_{\sigma_1,\sigma_2} c^*_{\sigma_1',\sigma_2'} |\sigma_1 \sigma_2\rangle \langle \sigma_1' \sigma_2'| ,$$

where * denotes complex conjugation. The *partial transpose* $\hat{\rho}^{T_A}$ of the density operator with respect to the subsystem of the qubit A is defined by interchanging the states $\sigma_1 \leftrightarrow \sigma_1'$ of qubit A in the above expression,

$$\hat{\rho}^{T_A} := \sum_{\substack{\sigma_1, \sigma_2 \\ \sigma_1', \sigma_2' \\ \in \{0,1\}}} c_{\sigma_1,\sigma_2} c^*_{\sigma_1',\sigma_2'} |\sigma_1' \sigma_2\rangle \langle \sigma_1 \sigma_2'|$$

and leaving the coefficients c_{ij} unchanged. The *negativity* $\mathcal{N}(\hat{\rho})$ of the density operator $\hat{\rho}$ is defined by

$$\mathcal{N}(\hat{\rho}) = \sum_{\lambda(\hat{\rho}^{T_A})} \frac{|\lambda| - \lambda}{2},$$

where the summation goes over all four eigenvalues λ of $\hat{\rho}^{T_A}$.

Now, consider the two-qubit state $|\psi\rangle = \alpha |00\rangle + \beta |11\rangle$ with yet arbitrary coefficients $\alpha, \beta \in \mathbb{R}$ (they could be $\in \mathbb{C}$, but for simplicity we restrict here to $\mathbb{R}$). For $|\alpha| = |\beta| = 1/\sqrt{2}$ this state is said to be maximally entangled since measuring, let's say, qubit A to be in state either $|0\rangle$ or $|1\rangle$ is equally likely with probability 1/2 (so maximally random), but having measured qubit B to be in state $|0\rangle$ ($|1\rangle$) sets qubit A also in state $|0\rangle$ ($|1\rangle$).

(d) Show that the partially transposed density operator $\hat{\rho}^{T_A}$ is given by the matrix

$$\rho = \begin{pmatrix} \alpha^2 & 0 & 0 & \\ 0 & 0 & \alpha\beta & 0 \\ 0 & \alpha\beta & 0 & 0 \\ 0 & 0 & 0 & \beta^2 \end{pmatrix}. \tag{3}$$

(3.5 points)

(e) Calculate the eigenvalues of the above density matrix (3) and show that the negativity $\mathcal{N}(\hat{\rho})$ is indeed maximal for $|\alpha| = |\beta| = 1/\sqrt{2}$. *(5.5 points)*