Introduction to quantum information processing
Exercise sheet 2

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***Note**: You may hand in your solutions in a group with up to three persons. Please provide your name to your solutions.*

Exercise 1: Euler angles (*10 points*)

(a) Show that, up to a meaningless global phase, any unitary operation on a single qubit can be written as $\hat{R}_z(\gamma)\hat{R}_x(\beta)R_z(\alpha)$, where $\hat{R}_i(\phi) = \exp\left(\mathrm{i}(\phi/2)\sigma_i\right)$ represents a rotation about the i-axis by angle $\phi/2$. (*5 points*)

(b) Explicitly write out this decompostion for the Hadamard gate. (*5 points*)

Exercise 2: Decomposition Part I (*10 points*)

Decompose the

(a) SWAP-gate (*5 points*)

(b) iSAWP-gate (*5 points*)

in CNOTs and single-qubit rotations, following the scheme we saw in the lecture.

Exercise 3: Decomposition Part II (*8 points*)

Build a CNOT-gate from an iSAWP-gate and single-qubit rotations.

Exercise 4: Deutsch-Josza algorithm and implementation of oracles (*12 points*)

The Deutsch-Josza algorithm was introduced in lecture, and we saw that the state of the the n input and the one output qubit is given by

$$|\psi\rangle = \sum_{x,z\in\{0,1\}^n} \frac{(-1)^{x\cdot z + f(x)}\,|z\rangle}{2^n}\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right], \tag{1}$$

where $x \cdot z = x_1 z_1 + x_2 z_2 + \cdots + x_n z_n$.

(a) Write out the term $|z\rangle = |00\ldots0\rangle$ in Eq. (1) for the n input qubits explicitly, and show that it is equal to ±1 for a constant function depending on whether $f \equiv 0$ or 1. Calculate this term for a balanced function as well, and explain why the measurment at the end is sufficient to check whether f is constant or balanced. Is it also possible to check if f is neither balanced nor constant with a single measurement of the n input qubits at the end as well? Explain shortly! (*4 points*)

(b) Now we want to build the oracle (gate U_f), which evaluates the function f for the n input qubits and sets the output qubit according to $U_f\,|x\rangle\,|y\rangle = |x\rangle\,|y \oplus f(x)\rangle$. For simplicity, we restrict outselves to a single input qubit, so that there are only two different possibilities for a balanced function or a constant one:

(i) $f(x) = x$,
(ii) $f(x) = 1 - x$,
(iii) $f(x) = 0$,

(iv) $f(x) = 1$.

For each of the above functions, give the action of the corresponding oracle in form of a table,

$\vert x\rangle \vert y\rangle$	$U_f \vert x\rangle \vert y\rangle$
$\vert 00\rangle$	?
$\vdots$	$\vdots$

write down the matrix representation and build the smallest possible circuit with the same action consisting only of CNOTs and single-qubit rotations. *(4 × 2 points)*