# Introduction to quantum information processing
## Exercise sheet 5

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using ***Note:*** *You may hand in your solutions in a group with up to three persons. Please provide your name to your solutions.*

## Exercise 1: Berry's phase (*10 points*)

What happens if a quantum system undergoes an adiabatic evolution, i.e. at each instant of time the quantum state is proportional to the instantaneous eigenstate it started in, and the final Hamiltonian coincides with the initial one? The initial and final states will of course be proportional (as the whole evolution was adiabatic), but they will in general differ by a phase. This phase, however, is not only of dynamical nature (you will see during the exercise what that means), but it has also a *geometric* contribution. This geometric phase is known as Berry's phase, and in this exercise we are going to investigate on its properties. It turns out that it allows for implementing fault-tolerant gates and holonomic quantum computing, and it is also important for understanding topological states of matter - that is not in the scope of this lecture, but Berry's phase is obviously an interesting subject.
Assume the Hamiltonian $H$ describing the system is controlled by some external parameters denoted by $\mathbf{R}$. (At the end, we will consider a particle in a magnetic field $\mathbf{B}$, whose three components are these parameters in that case.) For each value $\mathbf{R}$ of the external parameters, the Hamiltonian $H(\mathbf{R})$ has eigenvalues $E_n(\mathbf{R})$ and eigenstates $|n(\mathbf{R})\rangle$ satisfying

$$H(\mathbf{R})\,|n(\mathbf{R})\rangle = E_n(\mathbf{R})\,|n(\mathbf{R})\rangle \;\;, n = 0, \ldots . \tag{1}$$

If the parameters are controlled time-dependently, $\mathbf{R} = \mathbf{R}(t)$, the Hamiltonian and also its eigenenergies and eigenstates will depend on time as well, $H(t) = H(\mathbf{R}(t)), E_n = E_n(t), |n(t)\rangle = |n(\mathbf{R}(t))\rangle$. The time-dependent eigenstates $|n(t)\rangle$ are also called instantaneous eigenstates, as they still fulfill the eigenvalueproblem (1). Note however, that they do not fulfill the time-dependent Schrödinger equation

$$i\hbar \frac{d\,|\Psi(t)\rangle}{dt} = H\,|\Psi(t)\rangle\,. \tag{2}$$

Here, all dependencies of $H$ on external parameters and on time are not explicitly written out. As the instantaneous eigenstates form an orthonormal basis at each instant of time, we can express the general solution to the time-dependent Schrödinger equation as $|\Psi(t)\rangle = \sum_n c_n(t)\,|n(t)\rangle$, and plugging this ansatz into Eq. (2) gives

$$i\hbar \dot{c}_n(t) = c_n(t)\,[E_n(t) - i\hbar\,\langle n(t)|\dot{n}(t)\rangle] - i\hbar \sum_{k\neq n} c_k(t)\,\langle n(t)|\dot{k}(t)\rangle \;\;, n = 0, \ldots . \tag{3}$$

Solving these coupled differential equations and writing a general solution for the coefficients $c(t)$ is very hard. But we can significantly simplify this, assumed that the requirements for the adiabatic theorem are fulfilled so that the evolution is adiabatic. In that case, if we start in the $m$th eigenstate, $|\Psi(0)\rangle = |m(0)\rangle$ or equivalently $c_n(0) = \delta_{mn}$, we can ignore the coupling to all other eigenstates, which is described by the last sum in Eq. (3), and we are left with the equation

$$i\hbar \dot{c}_m(t) = c_m(t)\,[E_m(t) - i\hbar\,\langle m(t)|\dot{m}(t)\rangle]\,. \tag{4}$$

and all other coefficients stay zero all the time, so that $|\Psi(t)\rangle$ is always proportional to $|m(t)\rangle$.

(a) Show that

$$c_m(t) = \mathrm{e}^{i\theta_m(t)} \exp\left\{ -\int_0^t \langle m(\tau)|\dot{m}(\tau)\rangle \, \mathrm{d}\tau \right\} \tag{5}$$

with

$$\theta_m(t) = -\frac{1}{\hbar}\int_0^t E_m(\tau)\mathrm{d}\tau$$

solves Eq. (4). *(1 point)*

(b) The last exponential in Eq. (5) looks like a decaying exponential, which would violate the normalization requirement $\langle\Psi(t)|\Psi(t)\rangle = 1$. Use the normalization of $|m(t)\rangle$ to show that $\langle m(t)|\dot{m}(t)\rangle$ is a purely imaginary quantity. Consequently, we can write the last exponential as $\mathrm{e}^{i\gamma_m}$ with a real quantity $\gamma_m = i\int_0^t \langle m(\tau)|\dot{m}(\tau)\mathrm{d}\tau\rangle$. *(1 point)*

(c) How does $c_m$ (and thus $|\Psi\rangle$) evolve in time for a time-*independent* Hamiltonian? This phase is called *dynamical* phase. *(1 point)*

(d) The second phase, $\gamma_m$, is called *geometrical* phase (for reasons that become clear shortly), and its nature is fundamentally different from the dynamical one's. Recall that the time dependence of $|m(t)\rangle$ stems from the time dependence of the parameters, $|m(t)\rangle = |m(\mathbf{R}(t))\rangle$, to show that the geometrical phase can be written as a line integral

$$\gamma_m = i\int_\Gamma \langle m(\mathbf{R})\,|\nabla_{\mathbf{R}} m(\mathbf{R})\,)\rangle \cdot \mathrm{d}\mathbf{R}, \tag{6}$$

where the integration is along the path $\Gamma$ in the parameter space which is traced by $\mathbf{R}(t)$ during the adiabatic evolution. While the dynamical phase $\theta_m(t)$ is truly time-dependent, the geometrical phase only depends on the path $\Gamma$, and not on the time duration of the evolution - hence its name. *(1 point)*

The $\nabla_{\mathbf{R}}$ operator in Eq. (6) for $\gamma_m$ acts on $|m(\mathbf{R})\rangle$ so that $\langle m(\mathbf{R})\,|\nabla_{\mathbf{R}} m(\mathbf{R})\,)\rangle$ is not a gradient field, and a line integral along a closed path does not vanish in general. Instead, if the system undergoes an adiabatic evolution by changing the external parameters sufficiently slowly and is brought back to its original parameters so that $\Gamma$ is not a straight line, the state will pick up a non-vanishing phase $\gamma_m$. This phase is called Berry phase. Also, the vector field $i\,\langle m(\mathbf{R})\,|\nabla_{\mathbf{R}} m(\mathbf{R})\,)\rangle$ is called Berry vector field $\mathcal{A}$.

(e) Now, let's get more explicit: Consider an electron which is fixed at some position and placed in a magnetic field $\mathbf{B}$. The electron interacts with the external magnetic field, and the Hamiltonian describing this system is given by

$$H = \mathbf{B}\cdot\vec{\sigma} = \sum_{i=1}^{3} B_i\sigma_i. \tag{7}$$

This Hamiltonian depends on the three components of the magnetic field which are the parameters in this case, $\mathbf{R} = \mathbf{B}$. For this special case $\mathbf{R} \in \mathbb{R}^3$ we can use Stoke's theorem (and some little more algebra) to write the line integral along a closed path $\Gamma$ as a surface integral over the area enclosed by this path, $\partial A = \Gamma$:

$$\gamma_m = i\int_A \mathcal{B}_m \cdot \mathrm{d}\mathbf{A}$$

with

$$\mathcal{B}_m = i\sum_{n\neq m} \frac{\langle m|\nabla_{\mathbf{R}} H|n\rangle \times \langle n|\nabla_{\mathbf{R}} H|m\rangle}{(E_n - E_m)^2}.$$

(i) Show that

$$|+\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\phi}|1\rangle \hat{=} \begin{pmatrix}\cos\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\phi}\end{pmatrix}$$

is an eigenstate of the above Hamiltonian with eigenvalue $+B$, where the magnetic field is written in sperical coordinates,

$$\mathbf{B} = B\begin{pmatrix}\cos(\phi)\sin(\theta)\\ \sin(\phi)\sin(\theta)\\ \cos(\theta)\end{pmatrix}.$$

Give the other eigenstate $|-\rangle$. (*2 points*)

(ii) Show that

$$\mathcal{B}_+ = -\frac{\mathbf{B}}{2B^3}.$$

(*2 points*)

(iii) Show that $\gamma_+ = -\Omega_\Gamma/2$, where $\Omega_\Gamma$ is the solid angle enclosed by the closed path from the origin of the coordinate. (*2 points*)

## Exercise 2: Adiabatic Deutsch-Jozsa algorithm (*10 points*)

Deutsch-Jozsa's problem can also be solved using an adiabatic, linear interpolation between two Hamiltonians instead of the circuit model. Here, we choose the initial Hamiltonian $H_0$ to be $H_0 = 1 - |\phi\rangle\langle\phi|$ with the uniform superposition state

$$|\phi\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle = |+\rangle_x^{\otimes n}.$$

Let the final Hamiltonian be $H_1 = 1 - |\psi_f\rangle\langle\psi_f|$ with

$$|\psi_f\rangle = \frac{\mu_f}{\sqrt{N/2}}\sum_{i=0}^{N/2-1}|2i\rangle + \frac{1-\mu_f}{\sqrt{N/2}}\sum_{i=0}^{N/2-1}|2i+1\rangle$$

and

$$\mu_f = \left|\frac{1}{N}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}\right|.$$

(a) Show that $H_0$ and $H_1$ do not commute, i.e. $[H_0, H_1] = H_0H_1 - H_1H_0 \neq 0$. Why is that important (not only for this particular adiabatic algorithm, but in general)? (*1 point*)

(b) Give $|\psi_f\rangle$ for a constant function $f$ and a balanced one. How does a final measurement of $|\psi_f\rangle$ in the computational basis reveal whether $f$ is constant or balanced? (*2 points*)

(c) The purpose of this algorithm is of pedagogical nature only: What does it require to prepare $|\psi_f\rangle$? (*1.5 points*)

Let us now investigate on the adiabatic run time of a linear interpolation $H(s) = (1-s)H_0 + sH_1$ using the adiabatic theorem.

(d) For this purpose it is convenient to first proof the following lemma: Let $|\alpha\rangle$ and $|\beta\rangle$ be two states in some $N$-dimensional Hilbert space, and let $H_\alpha = 1 - |\alpha\rangle\langle\alpha|, H_\beta = 1 - |\beta\rangle\langle\beta|$. For any convex combination $H_\eta = (1-\eta)H_\alpha + \eta H_\beta$, where $\eta \in [0,1]$, the ground state gap $\Delta(H_\eta) \geq |\langle\alpha|\beta\rangle|$. (*2 points*)
*Hint:* As a starting point, assume that you expanded $|\beta\rangle = a|\alpha\rangle + b|\alpha^\perp\rangle$ with $\langle\alpha|\alpha^\perp\rangle = 0$ and constructed an orthonormal basis from $|\alpha\rangle$ and $|\alpha^\perp\rangle$. Compute the matrix representation of $H_\eta$ in this basis and compute its eigenvalues.

(e) Show that $||H^{(1)}|| \leq 2, ||H^{(2)}|| = 0$ and apply the above lemma to show that the adiabatic run time is $O(1)$. (*2 points*)

(f) A general statement about adiabatic algorithms: In principle, there is always a nonzero probability $q$ to end up in an excited state instead of the ground state, so that the whole algorithm failed and must be executed again. Express the probability, that the algorithm failed for the first $r-1$ times and succeeded in the $r$th run, in terms of $q$, and from that compute the average number of runs $\langle r \rangle$. Write this in a Taylor expansion about $q = 0$ up to the first non-vanishing order in $q$. (*1.5 points*)