Introduction to quantum information processing
Exercise sheet 6

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Note: *You may hand in your solutions in a group with up to three persons. Please provide your name to your solutions.*

Exercise 1: Cliques (*18 points*)

Suppose we are given an undirected graph $G = (V, E)$ with vertices V and edges E between them. A clique in this graph is a subset $W \subseteq V$ with $|W| = K$ vertices, such that the subgraph (W, E_W) (E_W is the edge set E restricted to edges between the vertices in W) is a complete graph. (A graph is complete, if every vertex in this graph is connected to every other vertex in the graph.) We now wish to find Hamiltonians that help us answering the questions: (1) Is there a clique of size K in the graph, and (2) what is the largest clique in the graph? For that purpose, we place a qubit on each vertex $v \in V$ of the graph and put connections between the qubits according to the edges between the vertices, via which the qubits can interact with each other. Also, it is convenient to introduce the bit operator

$$x_i = \frac{1 - \sigma_i^z}{2}$$

that acts on the ith qubit and has eigenvectors $|0\rangle$ and $|1\rangle$ with eigenvalues 0 and 1, respectively.

(a) In order to check for the existence of a clique of size K, we could choose the Hamiltonian $H_K = H_A + H_B$ with

$$H_A = A\left(K - \sum_i x_i\right)^2$$

$$H_B = B\left(\frac{K(K-1)}{2} - \sum_{(uv)\in E} x_u x_v\right),$$

where $A, B > 0$ are constants, which we need to choose such that the ground state energy is zero if and only if a clique of size K exists.

(i) If there exists a clique of size K, then we could measure the energy $E = 0$. Explain why! (*3 points*)

(ii) Now, let's ensure that the ground state energy $E > 0$ if there is no such clique. Suppose there are n qubits in the state $|1\rangle$. What is the minimum possible value for the energy, $E_{\min}(n)$? From that, show that $A > KB$ is sufficient to guarantee that $E_{\min} > 0$. (*5 points*)

(iii) Can the vertices, which form the clique, be found from the ground state? If yes, how? (*1 point*)

(b) In order to find the largest clique (or one of them), we can modify the above Hamiltonian by adding N extra qubits, where extra-qubit i is in state $|1\rangle$ if the largest clique has size i, and is in state $|0\rangle$ otherwise. Let y_i be the bit operator that acts only on the exra-qubit i with the action $y_i |n\rangle = \delta_{1n} |n\rangle$, $n \in \{0, 1\}$, where δ_{1n} denotes the Kronecker delta,

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} .$$

Consider the Hamiltonian $H = H_A + H_B + H_C$ with

$$H_A = A\left(1 - \sum_{i=1}^{N} y_i\right)^2 + A\left(\sum_{i=1}^{N} i y_i - \sum_j x_j\right)^2$$
$$H_B = B\left[\frac{1}{2}\left(\sum_{i=1}^{N} i y_i\right)\left(-1 + \sum_{i=1}^{N} i y_i\right) - \sum_{(uv)\in E} x_u x_v\right].$$

(Let's define H_C at the end.) We want cliques to satisfy $E_A = E_B = 0$, and to be the only ground states.

(i) Explain shortly why H_A and H_B ensure that the all cliques give energies $E_A = E_B = 0$. What do we need to require on A and B so that negative energies are absent? (*5 points*)

(ii) We now know that all ground states are cliques. Give a Hamiltonian H_C which helps us find the largest clique. (*4 points*)

Exercise 2: A simple 3SAT problem (*6 points*)

Consider the formula $F = (x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z)$ with boolean variables x, y, z.

(a) Construct the corresponding graph and (*3 points*)

(b) give the Hamiltonian (*3 points*)

following the scheme we saw in lecture.

Exercise 3: Baker-Campbell-Hausdorff formula (*16 points*)

Given are two linear operators A, B, which fulfill $[A,[A,B]] = [B,[A,B]] = 0$, and a scalar $\lambda \in \mathbb{C}$.

(a) Show that

$$[B, \mathrm{e}^{-\lambda A}] = \lambda \mathrm{e}^{-\lambda A}[A, B].$$

(*8 points*)

(b) Show that

$$\mathrm{e}^{\lambda(A+B)} = \mathrm{e}^{\lambda A}\mathrm{e}^{\lambda B}\mathrm{e}^{-\frac{\lambda^2}{2}[A,B]}.$$

(*8 points*)