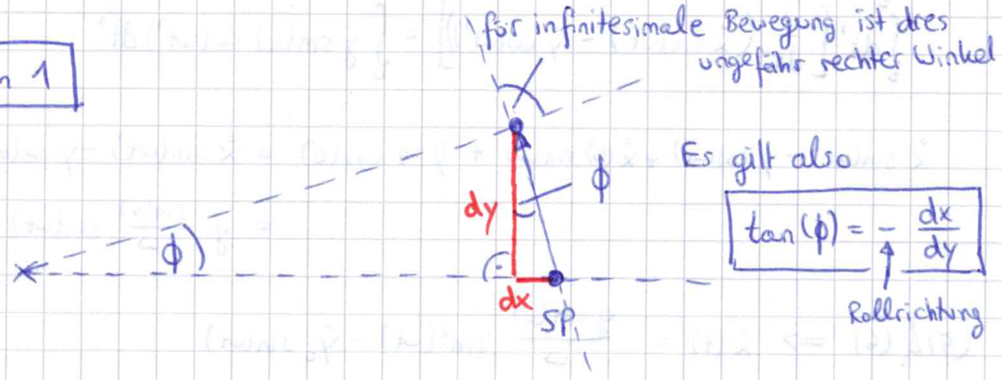
Saalübung, 22.07.2015

Problem 1

a)



Es gilt also

$$\boxed{\tan(\phi) = -\frac{dx}{dy}}$$

Rollrichtung

alternativ: $\cos(\phi)\,dx + \sin(\phi)\,dy = 0$ bzw. $\dot{x}\cos(\phi) + \dot{y}\sin(\phi) = 0$

b) $\mathcal{L} = T - U = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\dot{\phi}^2 - m \cdot g \cdot \sin(\alpha) \cdot y$

zyklische Variablen: x, ϕ $\Rightarrow$ Drehimpuls erhalten

c) Lagrange 1. Art: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \sum_{i=1}^{N} \mu_i a_{ik}$

für N Zwangsbedingungen $f_i = \sum_{k=1}^{Z} a_{ik}\,dq_k$ in differentieller Form

Lagrange-Gleichungen:

$x \longrightarrow \quad m\ddot{x} = \mu \cos(\phi) \qquad (1)$

$y \longrightarrow \quad m(\ddot{y} + g\sin(\alpha)) = \mu \sin(\phi) \qquad (2)$

$\phi \longrightarrow \quad I\ddot{\phi} = 0 \qquad (3)$

d) $\sin(\phi)\cdot(1) - \cos(\phi)\cdot(2) \Longleftrightarrow \ddot{x}\sin(\phi) - \ddot{y}\cos(\phi) - g\sin(\alpha)\cos(\phi) = 0 \qquad (4)$

zu lösende Glg.'en: $(3), (4)$, $\dot{x}\cos(\phi) + \dot{y}\sin(\phi) = 0 \quad (5)$

e) $\phi(0) = 0$, $\dot{\phi}(0) = \omega$, $x(0) = \dot{x}(0) = y(0) = 0$, $\dot{y}(0) = \dot{y}_0$

$\overset{(3)}{\Longrightarrow} \phi(t) = \omega t$

$$\begin{aligned}\frac{d}{dt}\left[\dot{x}\sin(\phi) - \dot{y}\cos(\phi)\right] &= \ddot{x}\sin(\omega t) + \dot{x}\omega\cos(\omega t) - \ddot{y}\cos(\omega t) + \dot{y}\omega\sin(\omega t) \\ &\overset{(5)}{=} \ddot{x}\sin(\omega t) - \ddot{y}\cos(\omega t) \\ &\overset{(4)}{=} g\sin(\alpha)\cos(\omega t)\end{aligned}$$

Integration beider Seiten:

$$\int_0^t dt' \left[\frac{d}{dt'}\left(\dot{x}\sin(\omega t') - \dot{y}\cos(\omega t')\right)\right] = \int_0^t g \sin(\alpha)\cos(\omega t')\, dt'$$

$$\left.\begin{aligned} \dot{x}\sin(\omega t) - \dot{y}\cos(\omega t) + \dot{x}(0)\sin(0) + \dot{y}(0)\cos(0) &= \dot{x}\sin(\omega t) - \dot{y}\cos(\omega t) + \dot{y}_0 \\ &= g\,\frac{\sin(\alpha)}{\omega}\,\sin(\omega t) \end{aligned}\right\} (6)$$

$$(5)\,\&\,(6) \Longrightarrow \dot{x}(t) = \frac{g\sin(\alpha)}{\omega}\sin^2(\omega t) - \dot{y}_0\sin(\omega t)$$

$$\dot{y}(t) = -\frac{g\sin(\alpha)}{\omega}\sin(\omega t)\cos(\omega t) + \dot{y}_0\cos(\omega t)$$

Integration mit $x(0) = y(0) = 0$:

$$\begin{cases} x(t) = \dfrac{g\sin(\alpha)}{4\omega^2}\left(2\omega t - \sin(2\omega t)\right) + \dfrac{\dot{y}_0}{\omega}\left(\cos(\omega t) - 1\right) \\[2ex] y(t) = \dfrac{g\sin(\alpha)}{4\omega^2}\left(\cos(2\omega t) - 1\right) + \dfrac{\dot{y}_0}{\omega}\sin(\omega t) \end{cases}$$