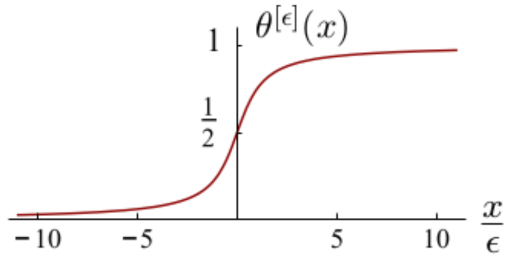
**Lösung Beispielaufgabe 1: Lorentz-Darstellung der Dirac-delta-Funktion [4]** (T0_0380a)

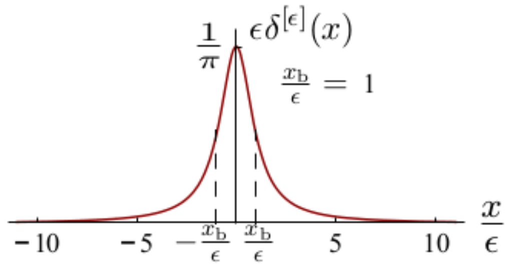
Wir müssen verifizieren, dass $\delta^{[\epsilon]}(x)$ im Limes $\epsilon \to 0$ die definierenden Eigenschaften der Dirac-delta-Funktion besitzt, nämlich:

(i) $\delta(0) = \infty\,,$ (ii) $\delta(x \neq 0) = 0\,,$ (iii) $\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x) = 1\,.$




Lorentz-Peak: $\delta^{[\epsilon]}(x) = \delta^{[\epsilon]}(x) = \dfrac{\epsilon/\pi}{x^2+\epsilon^2}\,.$

(i) Höhe: $\delta^{[\epsilon]}(0) = \dfrac{1}{\pi\epsilon} \xrightarrow{\epsilon\to 0} \boxed{\infty}\,.\checkmark$

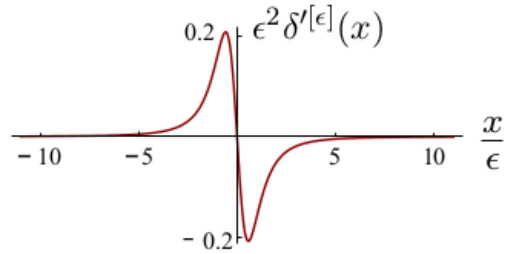



(ii) Breite: $\dfrac{1}{2} = \dfrac{\delta^{[\epsilon]}(x_\mathrm{b})}{\delta^{[\epsilon]}(0)} = \dfrac{\epsilon^2}{x_\mathrm{b}^2+\epsilon^2}$

$\Rightarrow \quad x_\mathrm{b} = \epsilon \xrightarrow{\epsilon\to 0} \boxed{0}\,.\checkmark$

$\delta^{[\epsilon]}(x\neq 0) \xrightarrow{\epsilon\ll x} \dfrac{\epsilon/\pi}{x^2}\left[1+\mathcal{O}(\epsilon^2/x^2)\right] \xrightarrow{\epsilon\to 0} \boxed{0}\,.\checkmark$

(iii) Gewicht, berechnet mittels Substitution:




$x = \epsilon\tan y = \epsilon\dfrac{\sin y}{\cos y}\,,$

$\dfrac{\mathrm{d}x}{\mathrm{d}y} = \dfrac{\epsilon}{\cos^2 y}\,, \quad x^2+\epsilon^2 = \dfrac{\epsilon^2}{\cos^2 y}\,.$

$$\int_{-\infty}^{\infty}\mathrm{d}x\,\delta^{[\epsilon]}(x) = \int_{-\infty}^{\infty}\mathrm{d}x\,\frac{\epsilon/\pi}{x^2+\epsilon^2} \overset{x=\epsilon\tan y}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}y\,\frac{\mathrm{d}x}{\mathrm{d}y}\frac{\epsilon/\pi}{\epsilon^2/\cos^2 y} = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}y \;= \boxed{1}\,.\checkmark$$

Somit besitzt $\delta^{[\epsilon]}(x)$ die definierenden Eigenschaften (i)-(iii) einer Dirac-$\delta$-Funktion. ✓

(iv) Stufe: $\displaystyle\theta^{[\epsilon]}(x) = \int_{-\infty}^{x}\mathrm{d}x'\delta^{[\epsilon]}(x') = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{y(x)}\mathrm{d}y' = \frac{1}{\pi}y'\Big|_{-\frac{\pi}{2}}^{y(x)} = \frac{1}{\pi}\left[\arctan(x/\epsilon)+\frac{\pi}{2}\right]$

$$= \boxed{\tfrac{1}{2}\left[\tfrac{2}{\pi}\arctan(x/\epsilon)+1\right]} \quad \xrightarrow{\epsilon\to 0} \begin{cases} 1 & \text{für} \quad x>0, \\ \frac{1}{2} & \text{für} \quad x=0, \\ 0 & \text{für} \quad x<0\,. \end{cases}$$

(v) Ableitung: $\delta'^{[\epsilon]}(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\epsilon/\pi}{x^2+\epsilon^2} = \boxed{-\dfrac{2x\epsilon/\pi}{(x^2+\epsilon^2)^2}}\,.$